An Introduction to Ramsey Theory

OUTLINE

- History
- Pigeonhole
- Ramsey Number
- Other Ramsey Theory
- Thinking about Ramsey

History

- Frank Plumpton Ramsey 1903–1930
- British mathematician
- Ramsey is a **BRANCH** of theory
- Ramsey theory ask: "how many elements of some structure must there be to guarantee that a particular property will hold?"

Pigeonhole

- Pigeonhole is a **simple** theory
  - m objects divide into n classes
  - at least [m/n] objects appears
- Application can be **subtle**
- Pigeonhole and Ramsey are closely linked
  - Some Ramsey can be proved by Pigeonhole
  - They both satisfy “how many elements can guarantee a property”

Ramsey Number

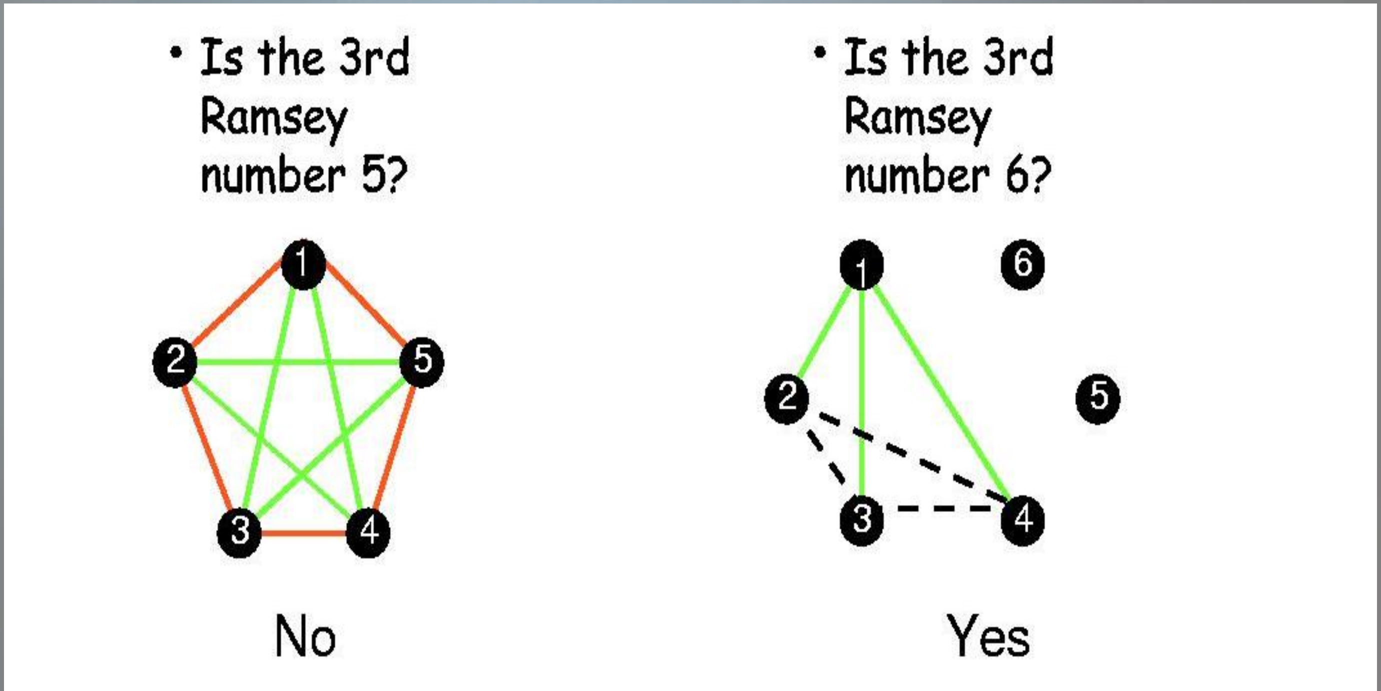
- Ramsey numbers part of mathematical field of graph theory
- k_m is defined as a graph containing *m* nodes and all possible line between the nodes
- Ramsey functions notated as $K(r, b)=n$
  - *K* is Ramsey function
  - *r, b* are independent variables
  - *n* is result of Ramsey function; called Ramsey number
- Ramsey function gives smallest graph size that when colored in any pattern of only two colors, will not contain sub-graphs of size *r* or *b* (i.e. does not contain a k_r or k_b)

Ramsey Number

- The Ramsey number R(m,n)gives the solution to the party problem, which asks the minimum number of guests R(m,n) that must be invited so that at least m will know each other or at least n will not know each other.
- R(X,1)=R(1,X)=1
- R(2,X)=X
- R(3,3)=??

Ramsey Number

Example: R(3,3) = 6



Possible to create a mapping without a k_3 in a 5-node graph (a k_5)

Not possible to create a coloring without a k_3 in a 6-node graph (a k_6) and this R(3,3) = 6

Ramsay Theorem

- In the language of graph theory, the Ramsey number is the minimum number of vertices v=R(m,n),such that all undirected simple graphs of order v contain a clique of order **m** *or* an independent set of order **n** .
- Ramsey theorem states that such a number exists for all m and n.

Existence Proof of R(r,s)

- for the 2-colour case, by induction on $r + s$
- Existance proof by proving a explicit bound
- base: for all n, $R(n, 1) = R(1, n) = 1$
- By the inductive hypothesis $R(r - 1, s)$ and $R(r, s - 1)$ exist.
- Claim: $R(r, s) \leq R(r - 1, s) + R(r, s - 1)$

Existence Proof of R(r,s)

- Consider a complete graph on $R(r-1, s) + R(r, s-1)$ vertices
- Pick a vertex v from the graph, and partition the remaining vertices into two sets M and N, such that for every vertex w, w is in M if (v, w) is blue, and w is in N if (v, w) is red.
- Because the graph has $R(r-1, s) + R(r, s-1) = |M| + |N| + 1$ vertices, it follows that either

$$|M| \geq R(r-1, s) \text{ or } |N| \geq R(r, s-1)$$

- In the former case, if M has a red K_s then so does the original graph and we are finished.
- Otherwise M has a blue K_{r-1} and so $M \cup \{v\}$ has blue K_r by definition of M. The latter case is analogous.
- R(4,3)=??

Small Ramsey Numbers

- $R(4,3) <= R(3,3) + R(4,2)$
  $<= 6 + 4 = 10$
- $R(4,3) = 9$

M	N	R(M,N)	Reference
3	3	6	Greenwood and Gleason 1955
3	4	9	Greenwood and Gleason 1955
3	9	36	Grinstead and Roberts 1982
3	23	[136, 275]	Wang et al. 1994
5	5	[43, 49]	Exoo 1989b, McKay and Radziszowski 1995
6	6	[102, 165]	Kalbfleisch 1965, Mackey 1994
19	19	[17885, 9075135299]	Luo et al. 2002

A generalized Ramsey number

- A generalized Ramsey number is written $r=R(M_1,M_2,\ldots,M_k\ ;\ n)$
- It is the smallest integer **r** such that, no matter how each n-element subset of an **r**-element sets is colored with **k** colors, there exists an **i** such that there is a subset of size $\mathbf{M_i}$, all of whose n-element subsets are color **i**.

A generalized Ramsey number

- $R(M_1,M_2,\ldots,M_k ; n)$
- when n>2, little is known.
  - R(4,4,3)=13
- When k>2, little is known.
  - R(3,3,3)=14
- Ramsey number tell us that R(m1,m2,…,mk;n) always exist!

Other Ramsey Theory

- Graph Ramsey Number
- Ramsey Polygon Number
- Ramsey of Bipartite graph
- ……

Graph Ramsey Number

- Given simple graphs G1,…,Gk,the graph Ramsey number R(G1,…,Gk) is the smallest integer n such that every k-coloring of E(Kn) contains a copy of Gi in color i for some i.

Extension to Hypergraphs

- For any integers m and c, and any integers $n_1,\ldots,n_c$, there is an integer $R(n_1,\ldots,n_c;c,m)$ such that if the hyperedges of a complete m-hypergraph of order $R(n_1,\ldots,n_c;c,m)$ are coloured with c different colours, then for some i between 1 and c, the hypergraph must contain a complete sub-m-hypergraph of order n_i whose hyperedges are all colour i.

Ramsey Theory Applications

- **Number Theory : Schur's theorem**
  if N *is partitioned into a finite number of classes, at least one partition class contains a solution to the equation x + y = z.*
- **Computational geometry:** *Erdos-Szekeres theorem*

$$2^{n-2} + 1 \leq g(n) \leq \binom{2n-4}{n-2}$$

where g(*n*) *denotes the smallest number such that any set of at least g*(*n*) *points in general position in the plane contains n points in convex position. The Erdos-Szekeres theorem is the consequence of the finite Ramsey theorem.*

Final Thoughts

- Results in Ramsey theory typically have two primary characteristics:
  - **non-constructive**: exist but non-consturctive
    - This is same for pigeonhole
  - **Grow exponetially:** results requires these objects to be enormously large.
    - That's why we still know small ramsey number
    - Computer is useless here!

Final Thoughts...

The reason behind such Ramsey-type results is that: “The largest partition class always contains the desired substructure”.

REFERENCES

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- http://en.wikipedia.org/wiki/Ramsey's_theorem
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- Ramsey Theory Applications:Vera Rosta

Questions

Thank you