The Probablistic Method

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1 Preliminaries

For definitions of graphs and hypergraphs see [1]. You will find [2, 3] useful for preliminaries about random variables and elementary probability theory, including concepts such as the subadditivity property of probabilities and the linearity of expectations. We will often use conditional probabilities in our constructions and analyses.

2 A simple result on hypergraph coloring using the first momemt method

We show that a hypergraph H with *fewer* than 2^{k-1} hyperedges (where each hyperedge has at least k vertices) is 2-colorable. Color the vertices at random giving colors with probability $\frac{1}{2}$. Each vertex is colored independently. Let A_e be the event that a certain hyperedge e is monochromatic. Since $Pr(A_e) \leq 2^{-(k-1)}$ and $|E(H)| < 2^{-(k-1)}$, the probability that at least one hyperedge is monochromatic is bounded by $|E(H)| \times 2^{-(k-1)} < 1$. [Here, $E(H)$ is the set of hyperedges of H.] Therefore, there is a coloring with no hyperedge monochromatic. We have used the subadditivity property of probability.

The same argument can also be placed differently if we use the *indicator variable* X_e, which is 1(0) if the hyperedge e is monochromatic (not monochromatic). Define $X = \Sigma_e X_e$. We find $E(X)$ as $\Sigma_e E(X_e)$. However, we know that $E(X_e) = Pr(A_e) \leq 2^{-(k-1)}$. So, $E(X) = |E(H)| \times 2^{-(k-1)} < 1$ because $|E(H)| < 2^{-(k-1)}$. Here, we use the property of linearity of expectations.

It is worthwhile noting that 2-colorability is easy for 2-uniform hypergraphs (ordinary graphs are easily checked for bipartiteness), and NP-complete, even for 3-uniform hypergraphs.

3 A simple Las Vegas algorithm for coloring *sparse* hypergraphs

We know now that a simple randomized coloring scheme results in proper coloring of the special class of hypergraphs, as in Section 2, with positive probability. So, we can assert the existence of a proper coloring. However, generating one such coloring is not quite easy. The exposition here is based on Sections 6.9.2 and 6.9.3 of [10]. Let us further restrict the number of edges to fewer than 2^{k-2}, where we also restrict the number of vertices in each hyperedge to exactly $k \geq 2$. We claim that in any random trial for coloring the vertices, some hyperedge remains monochromatic with probability at mosr $\frac{1}{2}$. [Hint: The probability that some hyperedge is monochromatic is less than $2^{k-2} \times 2^{-(k-1)} \leq \frac{1}{2}$.] So, on failure, we can repeat the random trial, and keep doing so, until we get a 2-coloring that is already guaranteed to exist. We will surely converge to a 2-coloring after several steps, with an average of 2 steps. Why?

4 Use of Lovasz's local lemma (LLL)

We may now withdraw the restriction limiting the cardinality of $E(H)$, the hyperedge set of H. So, we may no longer use the first moment method. However, we may restrict interaction between hyperedges by allowing a hyperedge e to intersect at most 2^{k-3} other hyperedges forming the set N_e, where $|N_e| \leq 2^{(k-3)}$. We know that $Pr(A_e) = p \leq 2^{-(k-1)}$ and also that A_e is mutually independent of all A_f where f is not a neighbor of e. [We omit this independence proof as in [2] but stress that it is non-trivial and necessary.] So, A_e depends on at most $d \leq 2^{(k-3)}$ similar events. Since $4pd \leq 1$ (see [3, 2] for complete proofs of LLL), we know by LLL (stated below) that there is a coloring with no monochromatic hyperedges.

Lemma 1 *Let $A_1, A_2, \ldots, A_k$ be a series of events such that each event occurs with probability at most p, and each event is independent of all the other events except for at most d of them. If $4pd \leq 1$, then there is a nonzero probability that none of the events $A_i, 1 \leq i \leq k$ occur.*

References

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