Advanced graph theory: Tutorial 2: CS60047 Autumn September 09, 2022

1. Suppose a graph G has no 1-factor. Show that there must be a vertex x such that each edge adjacent to x is in some maximum matching.

   [Let x be an unmatched vertex in a maximum matching F, and let y be a neighbour of x. If y is not covered by an edge e of F then $F + xy$ would be a larger matching. So, see $F - e + xy$ is also a maximum matching.]

2. Let G be a connected $2d$-regular graph with an even number of edges. Prove that G has a d-factor.

   [Consider an Euler trail L of G. Consider every second edge of the trail. This is possible as the number of edges is even. The selected edges for a d-factor.]

3. Show that the number of triangles in any simple graph of n vertices and m edges is at least $\frac{4m}{3n}(m - \frac{n^2}{4})$.

   [For any edge xy there are at least $d(x) + d(y) - n$ vertices adjacent to both x and y. So, this is also a lower bound on the number of triangles sitting on xy. A third of the sum of such estimates over all edges, lower bounds the number of triangles in the graph. This estimate is a third of $\sum(d(x))^2 - mn$, which is at least a third of n times the square of the average of vertex degrees minus mn by the Cauchy-Schwartz inequality.]

4. We wish to determine the maximum number $|E|$ of edges permissible so that a $K_{r,s}$ does not appear as a subgraph in a bipartite graph $G(V_1 \cup V_2, E)$. For all $x \in V_2$, (W, x) pairs must sum up to at most $(s-1)\binom{m}{r}$, which must cap $\binom{d(x)}{r}$ summed over all $x \in V_2$, where W is a subset of V_1 of r vertices connected to the same $x \in V_2$. This is necessary. For a solution see Theorem 9.5, Combinatorial Geometry by Pach and Agarwal, Wiley Interscience Series in Discrete Mathematics and Optimization. The upper bound sought is $c_{r,s}(mn^{1-\frac{1}{r}} + n)$, where the constant $c_{r,s}$ depends only on r and s.