Advanced graph theory: Homework 4: CS60047 Autumn 2022

Instructor: Sudebkumar Prasant Pal
Teaching assistants: Aayushi Vidyanta and Karnam Sai Keerthana

October 01, 2022

Advanced graph theory: Homework 4: CS60047

Autumn 2022

1. For any graph G show that there are at least $\binom{\chi(G)}{2}$ edges.

   [Hints: We know that $\chi(K_n) = n$ and $\chi(C_n)$ is 2 for n even and 3, otherwise, for all $n \geq 3$.]

2. Is it true that $\chi(G)+\chi(\bar{G}) \leq n+1$, where n is the number of vertices of the perfect graph G? Why? Does this hold for general graphs? Why?

   [Solution: Let G be a general n-vertex graph. Let all $(n-1)$-vertex graphs H be such that $\chi(H) + \chi(\bar{H}) \leq n$. Let v be any vertex of G. So, $H = G - v$ and $\bar{H} = \bar{G} - v$. Let the degree of v in G be d so that the degree of v in $\bar{G}$ is $n - d - 1$. Obviously, $\chi(G) \leq \chi(H) + 1$ and $\chi(\bar{G}) \leq \chi(\bar{H}) + 1$. Suppose both of these are equalities. Then, $\chi(H) \leq d$ and $\chi(\bar{H}) \leq n - d - 1$, whence $\chi(G) + \chi(\bar{G}) \leq n + 1$. Else, say wlog $\chi(G) < \chi(H) + 1$. Then $\chi(G) \leq \chi(H)$. So, $\chi(G) + \chi(\bar{G}) \leq \chi(H) + \chi(\bar{H}) + 1 \leq n + 1$.]

3. Suppose a simple graph has a quadrilateral then show that it has more than $\frac{n}{4}(1 + \sqrt{(4n-3)})$ edges.

   [Hints: For any two vertices x and y, we cannot have two common neighbours if we wish to exclude C_4 or a quadrilateral. So, an $\binom{n}{2}$ upper bound is immediate on the number of such pairs that have a common neighbour. A vertex z having two neighbours x and y can have more such pairs of neighbours if its degree $d(z)$ is more than two. You can use Jensen's inequality suitably.]

4. We say that a graph is *randomly traceable* if a spanning path always results upon starting at any vertex of G and then successively proceeding to any vertex not yet chosen until no new vertices are available. Show that an even graph G with $n > 4$ vertices exists, that is randomly traceable, but which is neither a C_n nor a K_n.

   We say that a graph is *arbitrarily traversable* from a vertex v_0, if starting a traversal at v_0, we traverse any incident edge, and on arriving at a vertex u, we depart from u by traversing any incident edge not yet used, and continue until no new edges remain. Show that if a graph G is arbitrarily traversable from a vertex v_0 then v_0 has maximum degree.

5. We wish to show that intersections of subtrees of a tree obey the Helly property, whereby the set $\mathcal{S} = \{T | T \in \mathcal{T}\}$ of pairwise intersecting subtrees of a tree $\mathcal{T}$ also has a non-empty intersection $\cap_{T\in\mathcal{S}} T$. Complete

the following argument for establishing this Helly property for intersecting subtrees of a tree.

[Hints: Suppose we use induction on the number k of subtrees of an n-vertex tree $\mathcal{T}$. Conisder k subtrees $T_1, T_2, ..., T_k$ of $\mathcal{T}$ which intersect pairwise. For the sake of contradiction we assume that they do not have a common intersection. However, by the induction hypothesis $T_1, T_2, ..., T_{k-1}$ intersect in say a subtree T_0. As T_k misses T_0 let us find a connecting path P from T_0 to T_k with a vertex $x \in P \cap T_k$ and a vertex y adjacent to x on P closer to T_0. Now $\mathcal{T} - xy$ has connected components where the edge xy separates T_0 from T_k.]

6. Show that a graph G is perfect if and only if it has the property that every induced subgraph H contains an independent set $A \subseteq V(H)$ such that $\omega(H - A) < \omega(H)$.

   Is the above property characterizing perfect graphs equivalent to the property "every induced subgraph H of G has an independent set meeting every clique of H of size $\omega(H)$".

   [Hint: Use induction.]

7. A graph G is such that in every induced subgraph H each maximal independent set of H meets every maximal complete subgraph of H. Show that such a graph G is perfect. Does the converse hold as well? Why?

   [Hint: Use induction.]

8. (Szekeres-Wilf (1968)) Show that for any graph G, $\chi(G) \leq 1+\max \delta(G')$, where the maximum is taken over all induced subgraphs of G. Also, show that $\chi(G) \leq n - \alpha(G) + 1$, where n is the number of vertices of G.

   [Hint: Let $\chi(G) = c \geq 2$. If H is any smallest induced subgraph such that $\chi(H) = c$, then show that for all induced subgraphs H' of H, we have $\max \delta(H') \geq \delta(H) \geq c - 1$.]

9. Show that an edge e in a graph G is a cut-edge if and only if e is contained in no cycle of G. (Theorem 2.3 in Bondy and Murty's textbook.)

10. Study exercise 1: Whitney's 1932 theorem on characterizing 2-connected graphs as those that have internally disjoint u, v-paths for every pair $\{u, v\}$ of vertices. (Theorem 3.2 from Bondy and Murty's textbook.)

    [Hint: Use induction on the length of the path or the non-trivial part, where Theorem 2.3 is used in the basis case. For the easier part, since there are two internally disjoint paths between u and v, dropping just one vertex cannot disconnect the graph. So, $\kappa(G) \geq 2$ implying G is 2-connected.]

11. Study exercise 2: Whitney's 1932 theorem on characterizing 2-connected graphs as those that have an ear decomposition. Show also that every cycle in a 2-connected graph is the initial cycle in some ear decomposition.

    [An *ear* of a graph G is a maximal path whose internal vertices have degree 2 in G. An *ear decomposition* of G is a decomposition $P_0, P_1, ..., P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an ear of $P_0 \cup P_1 ... \cup P_i$.]

12. Try Exercises 5.19, 5.21 and 5.22 from the textbook of Harary.