Advanced graph theory: Homework 2: CS60047 Autumn 2022

Instructor: Sudebkumar Prasant Pal
Teaching assistants: Aayushi Vidyanta and Karnam Sai Keerthana

August 18, 2022

Advanced graph theory: Homework 2: CS60047

Autumn 2022

1. The easier part of Kuratowski's theorem is to show that the presence of homeomorphs of K_5 or $K_{3,3}$ as subgraphs would make a graph non-planar. Prove this by (i) showing that K_5 and $K_{3,3}$ are non-planar, and (ii) the presence of a homeomorph of a non-planar graph causes non-planarity.

   [A graph G is a homeomorph of another graph H if G can be obtained by repeatedly adding degree-2 vertices w by deleting edge $\{u, v\}$, and adding edges $\{u, w\}$ and $\{w, v\}$. Note that H is planar if and only if its homeomorph G is planar.]

   [This amounts to showing the necessary condition that homeomorphs of none of the two Kuratowski's graphs can appear as subgraphs in a planar graph.]

   [The tougher part of Kuratwoski's theorem is to show that a graph is planar if it does not have subgraphs homeomorphic to the any of the two Kuratowski graphs.]

2. A connected simple planar graph with m edges, n vertices and girth g satisfies $m \leq \frac{g(n-2)}{g-2}$.

   [Hints: The dual of a planar embedding of a planar graph is such that the sum of degrees of the faces in the planar embedding is $2m$, exactly the same as the sum of degrees of the vertices. The degree of a face is the number of its bounding edges. So, $2m \geq gf$ where f is the number of faces. Now use Euler's equation $n + f = m + 2$. For $K_{3,3}$, $m = 9$, $g = 4$ and $n = 6$, this inequality is violated.]

3. The *thickness* of G is the least integer k so that G has *planar partition* $[G_1, G_2, ..., G_k]$. A *planar partition* of G is a collection $\mathcal{G} = [G_1, G_2, ..., G_k]$ of edge-disjoint spanning subgraphs of G, whose union is G. Derive a lower bound for the thichness $\theta(G)$ of G in terms of the number m of edges of G, the girth g of G, and the number of vertices n of G.

4. Apply Tutte's theorem to answer the following questions. Let $G' \neq K_n$ be an n-vertex simple connected undirected graph where adding any new edge e would introduce a perfect matching in $G' + e$, given that G' has no perfect matching. If S is the "bad" set as per Tutte's theorem whereby $o(G' - S) > |S|$, then show that (i) S induces a complete subgraph in G', (ii) each connected component of $G' - S$ also introduces a complete subgraph in G', and (iii) the vertices in S are connected to all the vertices in G'.

   [Hint: Suppose the edge $\{u, v\}$ is absent in G' where $u, v \in S$. Then adding this edge to G' introduces a perfect matching in $G' + \{u, v\}$ but does not change violated Tutte's condition $o(G' + \{u, v\} - S) = o(G - S) > |S|$, a contradiction. Similar arguments apply for the connected components of $G' - S$, and also to edges between S and the components of $G' - S$.]

5. Show that 3-regular graphs with no cut edges have a 1-factor.

6. Show that every regular graph of even degree has a 2-factor.

7. Show that every 3-regular graph with at most two cut edges has a 1-factor.

8. Show that every 2-connected 3-regular graph has a 1-factor.

9. For the bipartite graph $G(A \cup B, E)$, the subsets we consider are $X \subset A$, irrespective of whether the size of the neighbourhood $N(X) \subseteq B$ of X in G is equal to or greater than $|X|$. Here A is the set to be matched into B. Hall's condition requires $N(X)$ to be at least as big as X for every $X \subseteq A$, so that the whole of A may be covered by a matching. So, clearly, there are two cases, one of equality and the other of strictly being greater.

   We use induction to prove the hypothesis for matching the set A, given that the hypothesis holds for matching smaller sets that are subsets of A.

   We may have (i) $N(X)$ strictly larger than X for every $X \subset A$, $X \neq \phi$, or (ii) there is at least one $A_1 \subset A$, such that $N(A_1)$ is of the same size as A_1, $A_1 \neq \phi$. These are mutually exclusive and exhaustive cases. In either case, the induction hypothesis is that there is a matching that covers any proper subset of A. Using this assumption, we need to show that there is a matching that covers A.

   Work out the details of these two cases in order to show that satisfying the sufficiency condition for A implies the whole of A can be covered by a matching in G.