Advanced graph theory: CS60047: Autumn 2022

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- Transversal

19. Planarity
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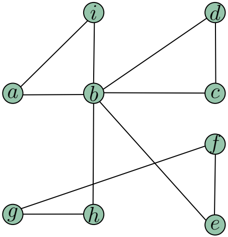
Walks and cycles

- A *subgraph* of a graph $G(V, E)$ with edge set E and vertex set V is a graph whose all edges and vertices are from E and V, respectively.
- A *spanning subgraph* of the graph $G(V, E)$ is a subgraph of $G(V, E)$ that has all the vertices in V.
- Given a graph $G(V, E)$, and a set $H \subseteq V$, the *induced subgraph* $G(H, E')$ is the maximal subgraph of $G(V, E)$ with the set of vertices in H.
- Every graph is its own induced subgraph.
- A *walk* is just a finite sequence of vertices where consecutive vertices are connected by an edge. So, vertices and edges can repeat. It is not difficult to show by induction that any closed walk of odd length at least three must contain an odd cycle. See Lemma 1.6.1 of [Sur10]. See Figure 1.

Walks and cycles (cont.)



Walk: $a, b, c, d, e, f, g, h, b, i, a$

Euler Tour: $b, i, a, b, c, d, b, e, f, g, h, b$

The vertex b has degree 6 and all the other vertices have degree 2

Figure: 1: Walks and cycles

Bipartite graph cycles

- We also note that a non-trivial graph is bipartite if and only if it has no odd cycles (Theorem 1.6.4 of [Sur10]).
- We can show the only if part by showing that every closed cycle $v_0 v_1 v_2 ... v_p = v_0$ will have an even size p.
- If V_1 and V_2 are the two partites then without loss of generality let us assume that $p_0 \in V_1$. Then $v_1, v_3, ... \in V_2$ and $v_0 = v_p, v_2, ... \in V_1$. Thus p is even.
- For the if-part, we assume that all cycles are even.
- From an arbitrary vertex $u \in V$, in the simply connected graph, we define sets V_1 (resp. V_2) of vertices of even (odd) distances from u.
- Now, suppose we have an edge connecting $v, w \in V_1$ then the shortest path from v to u appended by the shortest path from u to w and then the edge vw will form an odd cycle.

Vertex and edge connectivity

- *Edge connectivity* $\lambda(G)$ of a graph is the minimum number of edges whose removal results in a disconnected or trivial graph.
- So, $\lambda(G)$ is at most the minimum degree $\delta(G)$ in a simple connected graph G, because, by simply deleting as few as $\delta(G)$ edges we can disconnect the graph.
- Disconnecting means creating at least two components.
- A graph G on at least two vertices is k-edge-connected if any two vertices are connected by at least k edge-disjoint paths, and k-connected if any two vertices are connected by at least k internally-disjoint paths.
- So, for a k-edge-connected graph G, $\lambda(G) \geq k$.
- A graph on one vertex is defined to be both k-edge-connected and k-connected for $k = 0, 1$, but not for $k > 2$.

Vertex and edge connectivity (cont.)

- Thus every graph is 0-connected, a graph is 1-connected if and only if it is connected.
- A graph G is m-connected if the vertex connectivity $\kappa(G) \geq m$.
- Also, since internally-disjoint paths are edge-disjoint, k-connected graphs are k-edge-connected.
- Do we need to delete more than $\delta(G)$ vertices to disconnect a simple connected graph?
- The vertex connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph.
- We show that *vertex connectivity* $\kappa(G)$ is at most $\lambda(G)$. See Theorem 3.3.1 in [Sur10].
- For trivial or disconnected graphs both connectivities are zero.

Vertex and edge connectivity (cont.)

- If G is connected but has a cut edge e then $\lambda(G) = 1$, and additionally if $G = K_2$ then $\kappa(G) = 1$. Otherwise, at least one end of e has degree at least 2 and thus will be a cut vertex yielding $\kappa(G) = 1$.
- Now if $\lambda(G) \geq 2$, then after removing some $\lambda(G) - 1$ edges we must get a graph H that must have a cut edge, say $e = uv$.
- Since uv survives as a cut edge, in the connected graph H, we can now choose and delete one vertex (which is neither u nor v) from each of the $\lambda(G) - 1$ edges deleted.
- If the resulting graph is still connected then we can remove u or v additionally, thus disconnecting G with at most $\lambda(G)$ vertex deletions.

Vertex and edge connectivity (cont.)

- Study exercise 1: Whitney's 1932 theorem on characterizing 2-connected graphs as those that have internally disjoint u, v-paths for every pair $\{u, v\}$ of vertices. (Theorem 3.2 from Bondy and Murty's textbook [BM76].)
  [Hint: Use induction on the length of the path or the non-trivial part, where Theorem 2.3 [BM76] is used in the basis case. For the easier part, since there are two internally disjoint paths between u and v, dropping just one vertex cannot disconnect the graph. So, $\kappa(G) \geq 2$ implying G is 2-connected.]
- Study exercise 2: Whitney's 1932 theorem on characterizing 2-connected graphs as those that have an ear decomposition.
  [See Definition 4.2.7 and Theorem 4.2.8 in [Wes00].]
- Try Exercises 5.21 and 5.22 from [Har69].
- Theorem 3.2 in [BM76] can be generalized to k-connected graphs as a version of Menger's theorem as follows.

Vertex and edge connectivity (cont.)

- A graph with at least $k+1$ vertices (why $k+1$?) is k-connected if and only if any pair of distinct vertices have at least k vertex disjoint paths connecting them.
- The edge version of Menger's theorem states that a graph is k-edge-connected if and only if any pair of distinct vertices have at least k edge disjoint paths connecting them.
- Try exercises 3.2.1, 3.2.2, 3.2.3 and 3.2.4 from [BM76].

Paths and connectivity in trees

- Problem 6.24 in [Lov93] requires showing the existence of $n-k$ distinct paths of length k in a tree T with diameter $2k-3$.
- We can take the longest path P of $2k-2$ vertices $x_1, x_2, ..., x_{2k-2}$ in T, and consider distinct paths of length k from x_1 to x_{k+1}, x_2 to x_{k+2}, ..., and from x_{k-2} to x_{2k-2}. These are $k-2$ distinct paths in P.
- We can also find $n-(2k-2)$ distinct paths in T of length k, starting at each of the $n-(2k-2)$ vertices outside the diameter path P. This makes a total of $n-k$ paths.

Connectivity of the complement graph

- We know that undirected graphs have edges and therefore there may exist paths connecting vertices.
- In case there is no path connecting two arbitrary vertices u and v in an undirected simple graph G, the complement graph G' will contain the edge $\{u, v\}$ if u and v are not connected by an edge in G.
- However, if there is an edge between u and v in G, then these two vertices will not be directly connected in G'. Note that even in this case, will the two vertices be connected by a path in G'?

Connectivity of the complement ...

- So, we ask whether the complement of a simple disconnected graph must be connected.
- Let G be a simple disconnected graph and $u, v \in V(G)$. If u and v belong to different components of G, then clearly the edge $uv \in G'$, yielding a trivial path connecting the two vertices.

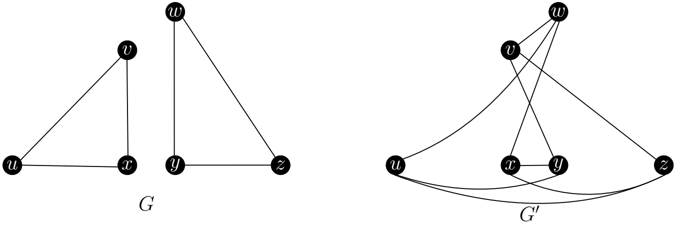


Figure: 2: Connectivity of the complement graph

Connectivity of the complement ... (cont.)

- If u and v belong to the same component of G but are not connected by an edge then we have a similar trivial path between them in the complement graph G'. If $uv \in G$, choose a vertex w in any other (disconnected) component of G. The edges uw and wv belong to G'. See Figure 2.

Degrees and connectivity

- It is interesting to see what happens if both the degree of each vertex as well as the girth of the graph, are both $k > 3$.
- In this case we show that there would be at least $2k$ vertices in the graph.
- For a vertex v, let K be the set of k neighbours of v.
- Take a vertex $w \in K$.
- If $x \neq v$ is a neighbour of w then x cannot be in the set K because that would yield a triangle, contradicting the fact that the girth is $k \geq 3$.
- So, all the $k-1$ neighhbours of w (other than v) are none of the vertices in K.
- Therefore, we already have $|K| + 1 + (k-1) = 2k$ vertices in the graph.

Girth four regular graphs

- If we have high vertex degrees then we will have at least a proportionate number of edges even with bounded girth like four.
- Suppose we have k-regular graph of girth four.
- Let u have the set $N(u)$ as its k neighbours.
- With the same reasoning as in the case of large girth, we can say that for $v, w \in N(u)$, vw is not an edge.
- So, for one $v \in N(u)$, its $k-1$ neighbours other than u are not in $N(u)$, already account for $|\{u\}| + |N(u)| + k - 1 = 2k$ vertices.
- Suppose above $N(u) = \{v_1, v_2, ..., v_k\}$.
- Then we have edges $N(v_1) = \{u, w_1, w_2, ..., w_{k-1}\}$.
- We can connect each of $v_2, v_3, ..., v_k$ as well to $N(v_1) \setminus \{u\}$, whereas u is already connected to $N(u)$.
- This gives the complete bipartite graph with partites $N(u)$ and $N(v_1)$.

Distinct degrees

- If all the vertices have disinct degrees in a simple connected undirected graph then these must be 0, 1, 2, ..., and $n-1$, respectively.
- This implies that one vertex must be connected to all the $n-1$ other vertices, including the one with degree zero, a contradiction because we assumed the graph was connected.

High degree, longer path

- When there is much connectivity, like if each vertex has at least k neighbours, then we can also have long paths.
- If we take a maximal path starting at u, then all the k neighbours of u must be on this path because we cannot extend this maximal path by connecting a neighbour of u, thus requiring this maximal path to be of length at least k.
- See Proposition 1.2.28 in [Wes00].

Connectivity with edges

- Suppose we have an even graph. Can this graph have a cut edge?
- If it were so then dropping this edge would render two connected components in the disconnected graph to have odd degrees.
- However, no connected subgraph can have just a single odd degree vertex.
- Adding edges can increase connectivity; in other words, an edge added to a graph $G(V, E)$ may reduce the $c(G)$, that is, the number of connected components by at most 1.
- So, by induction we can show that $c(G)$ is lower bounded by $|V| - |E|$.
- A graph with two edges has exactly $|V| - 2$ connected components. A graph with one edge has exactly $|V| - 1$ components but a graph with three edges may have $|V| - 2$ or $|V| - 3$ components.

Degrees and independence

- If $\alpha(G)$ is the maximum size of an independent set in a simple graph $G(V, E)$ then each of the $|V| - \alpha(G)$ vertices have some edges coming out, such that the sum of degrees of these vertices is at least the number $|E|$ of edges of G.
- Thus $\Delta(G)(|V| - \alpha(G)) \geq |E|$, where $\Delta(G)$ is the maximum degree of a vertex in G.
- Thus we have $\alpha(G) \leq |V| - \frac{|E|}{\Delta(G)}$.
- In a regular graph $|E| = \frac{\Delta(G)}{2}|V|$, whence $\alpha(G) \leq \frac{|V|}{2}$.

Matchings and factors

- A *matching* M is an independent set of edges in a graph $G(V, E)$ [Die17].
- So, no vertex in the graph will be in more than one edge of a matching.
- A k-factor of G is a k-regular spanning subgraph of G.
- So, a subgraph is a 1-factor if and only if it is a matching for the entire set of vertices in the graph, or in other words, it is a *perfect matching*.
- Such perfectly matched graphs must therefore have an even number of vertices.
- Note that even non-bipartite graphs may have perfect matchings.
- We can characterize general graphs that have a perfect matching by Tutte's condition, as per Tutte's theorem (Theorem 3.3.3 in [Wes00]).

Berge's theorem

- A matching M in a graph G is a maximum matching in G if and only if G has no M-augmenting path. This is a result from 1957.
- Let p denote the statement "M is a maximum matching", and q denote the statement "there is no M-augmenting path".
- Then the statement p if and only if q has two implications.
- The if-part is $q \implies p$, and the only-if part is $p \implies q$.
- To show the only-if part we show therefore that $\neg p \vee q$ holds or $p \wedge \neg q$ does not hold.
- Suppose a maximum matching M has an M-augmenting path.
- Then we can demonstrate a larger matching, contradicting the assumption that M is a maximum matching. This completes the only-if part.
- For the if-part we show $\neg q \vee p$ holds or $q \wedge \neg p$ does not hold.

Berge's theorem (cont.)

- So, to show the if-part we show the impossibility of M being not a maximum matching as well as that there are no M-augmenting paths.
- Assume that there is no M-augmenting path, but M is not a maximum matching. We show that this is impossible as follows.
- Let the maximum matching be M' and F be the symmetric difference betweem M and M'.
- Since $|M'| > |M|$, at least one component C of F must have more edges from M'.
- Since all cycles are of even length, and edges alternate between M and M' in F, C must therefore be a path and not a cycle in F.
- The two extreme edges in C must thus be from M', yielding an M-augmenting path.
- This completes the proof of the if-part of Berge's theorem

Berge's theorem (cont.)

- Now we formally state the definition of the symmetric difference of two matchings and study its properties.
- If M and M' are matchings, then $M \Delta M' = (M \cup M') \setminus (M \cap M')$.
- We show that every component of this symmetric difference of is a path or an even cycle.
- At most one edge of M and at most one edge of M' is incident on any vertex v.
- So maximum degree of any node in F is 2. So, components of F must be paths or cycles.
- Also, edges in a path or cycle will alternate between edges of $M \setminus M'$ and $M' \setminus M$.
- So all cycles must be even.

Proof of Hall's theorem using alternating paths

- For a bipartite graph $G(X \cup Y, E)$, suppose the neighbourhood $N(S)$ of any subset of the partite X is at least as larger as S, then we must show that there is a matching that covers the whole of X. This is known as Hall's theorem. (See Theorem 3.1.11 in [Wes00])
- Equivalently, we can establish sufficiency by demostrating the contrapositive; if the maximum matching M fails to match a vertex say $u \in X$ then we must demonstrate a subset S of X whose neighbourhood $N(S)$ is smaller than $|S|$.
- Towards this goal, we find the vertex subsets $S(resp., T)$ of $X(resp., Y)$ that are in M-alternating paths starting at the unmatched vertex u of X. Here $u \in S$ and $T = N(S)$.
- The unmatched vertex u cannot reach out to opposite side vertices outside M, as in that case u would match a vertex outside M and create a matching larger than M.

Proof of Hall's theorem using alternating paths (cont.)

- Suppose we show that M matches $S \setminus \{u\}$ to T. Then we would have shown that $N(S) = T$ has only $|S| - 1$ elements, violating Hall's condition (given that we had started with the assumption that the maximum matching M failed to match $u \in X$).
- Now we show how M matches the whole of $S \setminus \{u\}$ to T.
- Each vertex of $S \setminus \{u\}$ must be reached from a vertex of T in some M-alternating path via an edge of M.
- Also, M being a maximum matching, by Berge's theorem we do not have an M-augmenting path.
- So, the whole of T is saturated.
- Thus $T = N(S)$, with M defining the bijective mapping.
- Now as an application of Hall's theorem we can show that a k-regular bipartite graph has a perfect matching i.e., a 1-factor.

A proof of Hall's theorem using induction

- For the bipartite graph $G(A \cup B, E)$, the subsets we consider are $X \subseteq A$, irrespective of whether $N(X)$ is equal to or greater $|X|$, where A is the set to be matched. Throughout $N(X)$ is identical to $N_G(X)$, the neighbourhood set of X in G.
- Hall's condition requires $N(X)$ to be at least as big as X. So, clearly, there are two cases, one of equality and one of strictly being greater.
- We use induction to prove the hypothesis for matching the set A, given that the hypothesis holds for matching smaller sets that are subsets of A.
- We assume that (i) $N(X)$ is strictly larger than X for every $X \subset A$, $X \neq \phi$, or (ii) there is at least one $A_1 \subset A$, such that $N(A_1)$ is of the same size as that of A_1, $A_1 \neq \phi$. Naturally, for A we assume $N(A)$ is at least as big as A.
- These two are mutually exclusive and exhaustive cases.

A proof of Hall's theorem using induction (cont.)

- In either case, the induction hypothesis is that there is a matching that covers any proper subset of A.
- We need to show by using induction that there is a also a matching that covers A.
- In the first case we connect $x \in A$ to one of its adjacent vertices $y \in B$. Now see $G_1 = G - x - y$. We know that such a y exists.
- Let $X \subseteq A - \{x\}$. Assume $X \neq \phi$. Why?
- Clearly, $X \neq A$. Also, note that y may or may not be in $N(X)$.
- Since this case assumes $|N(X)| > |X|$, so $|N_{G_1}(X)| \geq N(X) - 1 \geq |X|$, as (i) $N_{G_1}(X)$ will miss out y only if $y \in N(X) = N_G(X)$, and (ii) $N(X) > |X|$.
- So, by the induction hypothesis, there is a matching F_1 covering $A - \{x\}$ in G_1, which along with (x, y) gives the matching covering the whole of A in G.

A proof of Hall's theorem using induction (cont.)

- In the second case, $A_1 \subset A$, $A_1 \neq \phi$, and $|N(A_1)| = |A_1|$.
- So let G_1 be the subgraph of G induced by $A_1 \cup N(A_1)$, and let $G_2 = G - A_1 - N(A_1)$.
- Now in G_1, let $X \subseteq A_1$.
- Then $N_{G_1}(X) = N_G(X) = N(X)$, and $|N_{G_1}(X)| = |N(X)| \geq |X|$. This holds for every $X \subseteq A_1$.
- Thus by the induction hypothesis there is a matching F_1 in G_1 which matches A_1 with $N(A_1)$.
- In G_2, let $X \subseteq A - A_1$.
- Then $|N_{G_2}(X)| = |N(X \cup A_1)| - |N(A_1)| \geq |X \cup A_1| - |N(A_1)| = |X \cup A_1| - |A_1| = |X|$.
- So in G_2, by the induction hypothesis, we have a matching F_2 that matches $A - A_1$ with certain vertices in $B - N(A_1)$.
- $F_1 \cup F_2$ is the matching for A.

Matching by augmenting paths in bipartite graphs

- $G(X, Y, E)$ has vertex partites X and Y, and edges E. Let $M \subseteq E$ be a matching in G.
- We maintain $S \subseteq X$ (initially $S = U$, where U is the set of M-unsaturated vertices of X) and $T \subseteq Y$ (initially ϕ), and mark vertices in S one by one, stopping when all vertices in S are marked.
- An M-augmenting path from any unmarked $x \in S$ to a vertex $y \in N(x)$, if $xy \notin M$ (and thus y is unsaturated) is xy itself !
- If y is not unsaturated, and thus matched to $w \in X$ by M, then put y in T, as y is reachable from x, and put w in S, as it is reached from y.
- Once all such edges sitting on x are explored thus, mark x, and proceed iterating.
- See Theorem 3.2.2 [Wes00] and Algorithm 3.2.1.
- Note that M gets augmented until all vertices in S are marked.

Matching by augmenting paths in bipartite graphs (cont.)

- Two things now needed are to show are that on termination, M is the maximum matching with the same number of edges as the set $R = T \cup (X \setminus S)$, which is the (minimum) vertex cover.
- When we have the augmenting path, we enhance the matching, else we have already computed the maximum matching by Berge's theorem, and must now have the minimum vertex cover, thereby establishing the Konig-Egervary theorem.
- To show that R is indeed a vertex cover, we may show that S has no edges to vertices in $Y \setminus T$.
- We know that each vertex of $S \setminus U$ is matched by an edge of M to some vertex in T, and no vertex of S has an edge of M into $Y \setminus T$.
- Also, no non-M edges connect from S to $Y \setminus T$, as then we would have augmented M, extending T, a contradiction.

Matching by augmenting paths in bipartite graphs (cont.)

- So, the vertex cover is only T from Y and the whole of X, leaving out S from X, as not only $S \setminus U$ but also all of $U \subseteq S$ have been marked as having an edge into T.
- So, to cover edges from S, we include T in the vertex cover and to cover the edges from U, the remaining vertices in X, we include $X \setminus S$ in the vertex cover.
- Now T has only saturated vertices, and all vertices in T are matched to an equal number of vertices in S.
- The additional matching edges in M beyond the $|T|$ already mentioned are from S but not in U or those matched to T from S. These are thus the remnants in X after dropping S.
- This is thus a constructive proof of the Konig-Egervary theorem.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17]

- The theorem of Konig-Egervary is a well-known duality result stating that the size of the maximum matching is the same as the size of the minimum vertex cover in a bipartite graph.
- Let $A' \subseteq A$ and $B' \subseteq B$ be the two mutually disjoint subsets of V constituting the minimum vertex cover U for $G(V, E)$.
- Consider $A \setminus A'$ and $B \setminus B'$.
- These sets do not induce any edges in G and therefore constitute a maximum independent set, because $A' \cup B$' is the minimum vertex cover.
- So, $|N(A \setminus A')| \leq |B'|$.
- Now let us now assume that G does not have a matching for the whole of A, implying $|A'| + |B'| = |U| < |A|$, or $|A| - |A'| > |B'|$, and thus $|A \setminus A'| > |B'| \geq |N(A \setminus A')|$.

Proof of Hall's theorem using the Konig-Egervary theorem [Die17] (cont.)

- This establishes the contrapositive for the sufficiency condition for Hall's theorem with the subset $A \setminus A'$ as witness.
- Here, the strict inequality $|U| < |A|$ holds because the maximum matching size is the same as the size $|U|$ of the minimum vertex cover by the Konig-Egervary theorem, and at least one vertex in A is not matched in any maximum matching.

Notations and definitions about independence and covering

- For the sake of some notation, let us use
  $\alpha(G)$ to denote the size of the maximum independent (stable) set in a simple connected graph $G(V, E)$,
  $\beta(G)$ to denote the size of the minimum vertex cover,
  $\alpha'(G)$ for the size of the maximum matching, and
  $\beta'(G)$ for the size of the minimum edge cover.
- We know that $\alpha(G) + \beta(G) = |V| = n$ for any graph.
- For bipartite graphs we know by the Konig-Egervary theorem that $\beta(G) = \alpha'(G)$.
- For general graphs $\beta(G) \geq \alpha'(G)$ because we need to cover each edge of a matching by at least one vertex.
- We also know that for any graph, no edge can cover two vertices of an independent set.

Notations and definitions about independence and covering (cont.)

- So, we can write $\beta'(G) \geq \alpha(G)$.
- Further, note that by Gallai's theorem we know that $\alpha'(G) + \beta'(G) = |V| = n$ for any connected graph.
- To show that $\alpha(G) + \beta(G) = |V| = n$ for any connected graph, we argue as follows.
- If T is an independent set, then edges can have at most one endpoint in T.
- So each edge has at least one endpoint in $V \setminus T$, making it a vertex cover.
- Also, if $V \setminus T$ is a vertex cover, T will not have both endpoints of any edge.
- Study exercise: Proof of Gallai's theorem.

Applications of Hall's theorem

- We show using Hall's theorem that any simple undirected $2k$-regular bipartite graph has a 2-factor (cycle cover).
- See Theorem 3.3.9 in [Wes00] and Theorem 7.2.8 in [Jun99].
- The common property of a 2-factor and an Euler tour is that both span the graph, albeit in different ways.
- Whereas the 2-factor spans all the vertices in a subgraph where all vertices use only two edges, the Euler tour spans all the edges, each edge exactly once.
- Let us assume that the simple undirected $2k$-regular undirected bipartite graph G is connected.
- Let m be the number of its edges. Let v_0 be any vertex.
- Then we have an Euler tour $v_0, e_1, e_2, \ldots, e_m, v_m = v_0$, where $e_i = v_i v_{i+1}$.
- So, $e_1 = v_0 v_1$, $e_2 = v_1 v_2$, ..., $e_m = v_{m-1} v_m = v_{m-1} v_0$.

Applications of Hall's theorem (cont.)

- We replace each vertex v by two vertices v' and v'' and each edge $e_i = v_i v_{i+1}$ is replaced by $v'_i v''_{i+1}$.
- So, e_1 is replaced by $v'_0 v''_1$, going from the left to right, and e_2 is replaced by $v'_1 v''_2$, going from right to left.
- Due to the Euler tour, each original vertex $v \in V$, now has exactly k edges going left to right (right to left) from v', and exactly k edges going right to left (left to right) from v'' if v is on the left (right) set.
- The new graph is k-regular and has therefore a 1-factor; merging the split vertices back gives the 2-factor for the original graph G.
- This solution is from [Die17].

Proof of Gallai's theorem

- If we have a graph with no isolated vertices then we can show that $\beta'(G) \leq |V| - \alpha'(G)$.
- The trick is that we can start with a maximum matching M and generate an edge cover L of size $|V| - |M|$.
- The edge cover L of size $|V| - \alpha'(G)$ is at least the size $\beta'(G)$ of the minimum edge cover.
- How do we do this construction? We add to M one edge incident on each vertex uncovered by M.
- Since M covers $2\alpha'(G)$ vertices, the new edges added are only $|V| - 2\alpha'(G)$ in number.
- These edges along with the $\alpha'(G)$ edges of M form the edge cover L.
- To complete the proof of Galai's theorem, we must now show $\beta'(G) \geq |V| - \alpha'(G)$, or equivalently, $\alpha'(G) \geq |V| - \beta'(G)$.

Proof of Gallai's theorem (cont.)

- We start with a minimum edge cover L of size $\beta'(G)$, and construct a matching of size $|V| - \beta'(G)$, which has to naturally be of size no more than $\alpha'(G)$.
- This is so as L is a collection of k stars, with each star giving only exactly one edge to the matching M.
- The matching size is the number $k = |V| - |L| = |V| - \beta'(G)$ of stars counted by the central vertices of the stars because all edges of L end on peripheral vertices of their respective stars which number $|V| - k$.
- Why is L a collection of k stars?

Proving the Konig-Egervary theorem using Hall's theorem

- It is sufficient to show that for any minimum cardinality vertex cover Q of $G(X \cup Y, E)$, we can demonstrate a matching M of size $\beta(G) = |U|$. Why?
- (We know that $\beta(G) \geq \alpha'(G)$. We need at least as many vertices as the number of edges in the maximum matching in order to cover all edges.)
- Consider the partition of any minimum cardinality vertex cover Q into $R = Q \cap X$ and $T = Q \cap Y$.
- Consider (edge-disjoint) subgraphs H and H' induced by $R \cup (Y \setminus T)$ and $T \cup (X \setminus R)$.
- Using Hall's theorem we show that H has a matching for R into $Y \setminus T$ and H' has a matching for T into $X \setminus R$.
- So, a matching of size $|Q|$ from H and H' for the whole of G can be demonstrated.

Proving the Konig-Egervary theorem using Hall's theorem (cont.)

- Since $R \cup T$ is a vertex cover for G, no edges exist between $Y \setminus T$ and $X \setminus R$.
- For any $S \subseteq R$, consider $N_H(S) \subseteq Y \setminus T$. Can the vertex cover $R \cup T$ be replaced by $(R \setminus S) \cup N_H(S) \cup T$?
- Since this can never shrink the minimum vertex cover Q, we have Hall's condition $|N_H(S)| \geq |S|$ for any $S \subseteq R$.
- So, R matches into $Y \setminus T$ by Hall's theorem. See [Wes00].
- Similarly we can show that T matches into $X \setminus R$.

Proving the Konig-Egervary theorem using alternating paths [Die17]

- It is enough to show that there is a vertex cover whose size equals the size of the maximum matching in a bipartite graph.
- So, given a maximum matching M, we pick exactly one vertex from each edge of M and show that these vertices cover all edges of the graph.
- We do this using *alternating paths* [Die17]; later we also do this using the max-flow-min-cut theorem [GGL95].
- So take the two partite sets as A and B with the maximum matching M in bipartite graph $G(A \cup B, E)$.
- We define a set U as the collection of vertices of the edges of M, only one vertex per edge of M as follows, and show that U is a vertex cover.

Proving the Konig-Egervary theorem using alternating paths [Die17] (cont.)

- So, for each M-alternating path, ending with a vertex in B in the matching M, we take the end vertex in that path in U, which thus belongs to an edge e of M, provided just one vertex is taken from the edge e in M. For the remaining edges in M we take the vertex in A.
- Since by Berge's theorem, the graph has no M-augmenting path, each M-alternating path (starting at an M-unsaturated vertex in A) will end at a vertex in B belonging to an edge of M.
- Such a path could also contain just a single edge.
- We claim that U is a vertex cover. See [Die17]. At least all edges of M are covered as per the construction. Also, all other edges must be shown to be covered by U.

Proving the Konig-Egervary theorem using alternating paths [Die17] (cont.)

- For each M-unsaturated vertex $u \in A$ determine reachable vertices in B where M-alternating paths starting at u end. Add these vertices to U, and for other edges of M, add the ends in A on these edges of M to U.
- More formally, let $ab \in E$ be an edge with $a \in A$ and $b \in B$, $ab \notin M$. As all edges in M have at least one vertex in U, as per the construction/definition of U above, we now need to show that at least one of a or b is in U.
- Since ab is not in the maximum matching M, there must be an edge $a'b' \in M$ such that $a = a'$ or $b = b'$. Otherwise, we could add ab to M getting a bigger matching.
- But is it possible that a is unmatched? If so, then surely $b = b'$, as $a \neq a'$, and here ab is an alternating path of odd length unity, and so the end of $a'b' \in M$, chosen for U was the vertex $b' = b$.

Proving the Konig-Egervary theorem using alternating paths [Die17] (cont.)

- So, now we assume that a is matched, that is, $a = a'$ and $b \neq b'$. Now if $a' = a$ is not in U, then surely $b' \in U$, and some M-alternating path P ends in b'.
- But then there is also an M-alternating path P' ending in b as shown here.
- P may have b or may not have b.
- In the first case P' is Pb and in the other case P' is $Pb'a'b$.
- By the maximality of M, P' is not an M-augmenting path. Therefore, b must be matched, and was chosen for U from the edge of M containing it.
- So, ab is always covered by U.

Large number of edges lead to subgraphs with proportionate minimum vertex degree

- We observe that if G is a graph on n vertices with more than $(c-1)n$ edges, where c is a positive integer, then G has a subgraph H of minimum degree at least c (Lemma 7.1, page 74 [GGL95]).
- This is so as any minimal subgraph H with more than $(c-1)v(H)$ edges has the necessary property of minimum vertex degree at least c.
- If H had a vertex v of degree at most $c-1$, then subgraph $H \setminus \{v\}$ would contradict the choice of H because in that case $H \setminus \{v\}$, and not H would be the minimal subgraph with the required property.

Extremal results: Spanning subgraphs of high vertex degrees

- In similar vein, we can show that every graph G has a bipartite spanning subgraph B such that $degree_B(v) \geq \frac{degree_G(v)}{2}$ for all vertices v (Lemma 7.2, page 74 [GGL95]).
- We note that any bipartite spanning subgraph $B(X, Y)$ with the maximum number of edges has this property.
- Suppose B had a vertex v of degree less than $\frac{degree_G(v)}{2}$, and without loss of generality $v \in X$, then the bipartite spanning subgraph with bi-partition $(X \setminus \{v\}, Y \cup \{v\})$ would contradict the choice of B because this modified graph would have more edges.
- Such results may be required in the proofs of extremal properties where the number of edges is only of some modest smaller magnitude, serving a required purpose, even by restricting the class of graphs under consideration to bipartite graphs of large degree.

Extremal results: Spanning subgraphs of high vertex degrees (cont.)

- In the breadth-first search trees of bipartite graphs of large degree, the sets reachable grow rapidly.

Extremal results: Mantel's theorem

- We can put some weights (non-negative) for the vertices of the graph $G(V, E)$, so that the weights add up to 1, trying to maximize the sum over all edges of the products of weights assigned to its vertices.
- If all the weights are just $\frac{1}{n}$ then we have a sum $\frac{|E|}{n^2}$. This may not be the maximum though.
- We can show the maximum is attained when we assign $\frac{1}{2}$ to just two vertices connected as an edge in G, whereby the maximum sum is just $\frac{1}{4}$. So, $\frac{|E|}{n^2} \leq \frac{1}{4}$.
- For a pair $\{k, l\}$ of unconnected vertices, let x and y be the sum of weights assigned to the neighbours of vertices k and l respectively, where $x > y$. Let the weights assigned to vertices k and l be respectively z_k and z_l.
- We note that moving a small weight e from vertex l to the vertex k, will change the sum of products of weights of vertices joined by edges to $x(z_k + e) + y(z_l - e) \geq xz_k + yz_l$.

Bounding triangles in a graph

- We show that the number of triangles in any simple graph of n vertices and m edges is at least $\frac{4m}{3n}(m - \frac{n^2}{4})$.
- For any edge xy there are at least $d(x) + d(y) - n$ vertices adjacent to both x and y. Why? The remaining $n - 2$ vertices cover at least $d(x) + d(y) - 2$ edges. If c is the number of triangles sitting on base xy then $n - 2 + c \geq d(x) + d(y) - 2$ or $c \geq d(x) + d(y) - n$.
- So, this is also a lower bound on the number of triangles sitting on xy.
- However, due to counting thrice (once for each edge of every triangle), we consider only a third of the sum of such lower bound estimates over all edges as a lower bound for the number of triangles in the graph.
- This estimate is a third of $\sum(d(x))^2 - mn$, which is at least a third of n times the square of the average of vertex degrees minus mn by the Cauchy-Schwartz inequality. Why?

Bounding triangles in a graph (cont.)

- So, the number of triangles is at least

$$\frac{1}{3}\sum_{xy\in E}(d(x)+d(y)-n)$$

$$=\frac{1}{3}((\sum_{x\in V}d^2(x))-mn)$$

$$\geq\frac{1}{3}((n(\sum_{x\in V}\frac{d(x)}{n})^2)-mn)$$

$$=\frac{1}{3}(n(\frac{2m}{n})^2-mn)$$

- Now consider adding the squares of the degrees of all vertices in a triangle-free graph.
- View this summation over all vertices as a sum over all edges $xy\in E$.

Bounding triangles in a graph (cont.)

- For each edge xy we simple need to add $d(x)$ and $d(y)$, that is $d(x) + d(y) \leq n$, thereby adding $d(x)$ for each vertex $d(x)$ times.
- So the summation is simply mn as we have m edges.
- However, $(2m)^2 = (\Sigma_{x \in V} d(x))^2 \leq n \Sigma_{x \in V} d^2(x) = n \Sigma_{xy \in E}(d(x) + d(y)) \leq mn^2$, yielding $m \leq \frac{n^2}{4}$.

Turan's problem

- The $r = 2$ case of avoiding a K_{r+1} in a graph was attained with the maximum number $\frac{n^2}{4}$ of egdes by $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.
- We generalize now to the case for avoiding K_{r+1} for $r > 2$, again maximizing the number of edges in the n-vertex graph G
- To construct such a graph G of n vertices, we must note that G should have a K_r, otherwise, we could add an edge and still not have a K_{r+1}.
- Let A be the set of r vertices in some r-clique in $G(V, E)$.
- Let $B = V \setminus A$ be the set of the remaining $n - r$ vertices.
- Let $T_{n,r}$ be the n-vertex graph that is maximal, r-partite and balanced. Let $e(T_{n,r})$ be its number of edges.
- So, we have $|E| \leq \binom{r}{2} + (r-1)|B| + e(B) \leq \binom{r}{2} + (r-1)(n-r) + e(T_{n-r,r})$.

Turan's problem (cont.)

- The r-clique A has $\binom{r}{2}$ edges, each vertex in B can connect to at most $r-1$ vertices in A and by the induction hypothesis $e(B) \leq e(T_{n-r,r})$.
- To complete the proof, we need to argue that the final sum above is indeed $e(T_{n,r})$, which is easy to see; this part of establishing the equality is independent of A and B.
- Just drop r vertices, one from each partite.
- Exactly $n-r$ remaining vertices would each lose exactly $r-1$ neighbours.
- Question: Determine $e(T_{n,r})$ in terms of n and r.
- Guessing $e(T_{n,r}) \leq \frac{1}{2}(1-\frac{1}{r})(n^2)$, and by induction we claim that $e(T_{n,r}) \leq \binom{r}{2} + (r-1)(n-r) + e(T_{n-r,r}) \leq$ $\binom{r}{2} + (r-1)(n-r) + \frac{1}{2}(1-\frac{1}{r})(n-r)^2$.
- Question What is the maximum number of triples in an n-vertex 3-uniform hypergraph without a tetrahedron?

Turan's problem (cont.)

- If n is a multiple of r then $e(T_{n,r}) = \frac{1}{2}(1 - \frac{1}{r})(n^2)$.
- So, $r = 2$ gives us Mantel's theorem bound for missing a K_3. And $r = 3$ gives us the bound for missing a tetrahedron or K_4 as $\frac{n^2}{3}$.

A “Pigeonhole approach” to Turan’s problem

- Turan’s theorem in the “Pigeonhole principle mode” may be stated as follows.
- Let $G(V, E)$ be a graph with $n = mk$ vertices and more than $e = \binom{k}{2} m^2$ edges.
- Then G must have a K_{k+1}.
- We use induction on m to establish this result.
- Suppose G has no K_{k+1}.
- Then, the graph $G_1 = G - V(H)$ has $mk - k$ vertices which has no more than $k - 1$ edges connecting to H, where H is a K_k in G.
- So, G_1 must have more than $\binom{k}{2}(m)^2 - (k-1)(mk-k) - \binom{k}{2} = \binom{k}{2}(m-1)^2$ edges.
- This is so as the induced subgraph H had $\binom{k}{2}$ edges and each of the surviving $mk - k$ vertices in G_1 can connect to at most $k - 1$ vertices in H to make G.

A “Pigeonhole approach” to Turan’s problem (cont.)

- However, by the induction hypothesis, this means that G_1 with $k(m-1)$ vertices, and $\binom{k}{2}(m-1)^2$ contains a K_{k+1}, and hence, so does G, a contradiction to our assumption that G has no K_{k+1}.

Erdos-Stone theorem

- We will restate the result of Turan where we look for K_r-free graphs on n vertices, as stated in Theorem 9.3 in [PA11] for $r > 2$.
- The number of edges in such a graph G would never exceed the number of edges in $T_{r-1}(n)$, an $(r-1)$-partite graph that is *balanced*, *complete* and has no K_r.
- The equality here would hold if and only if $G = T_{r-1}(n)$.
- Each of the $r-1$ partites has either $\lceil \frac{n}{r-1} \rceil$ or $\lfloor \frac{n}{r-1} \rfloor$ vertices in $T_{r-1}(n)$.
- Exercise 9.5 in [PA11] gives the exact number of edges in $T_{r-1}(n)$ as $\frac{1}{2}(1-\frac{1}{r-1})(n^2-s^2)+\binom{s}{2}$, where $s = n \mod (r-1)$. These are quite dense graphs.
- Turan's theorem implies that a graph G with more than $e(T_{r-1}(n))$ edges would contain a K_r as a subgraph.

Erdos-Stone theorem (cont.)

- More edges would be required so that a graph G contains an r-partite complete subgraph K_r^t, that has t vertices in each partite. Here $K_r^t = T_r(rt)$, and t is any fixed constant.
- The Erdos-Stone theorem (Theorem 9.10 in [PA11]) states that K_r^t is contained in G if it has $n \geq n_0(r, t, \epsilon)$ vertices for any $\epsilon > 0$, and if we have at least $\frac{n^2}{2}(1 - \frac{1}{r-1} + \epsilon)$ edges in G.
- The number of edges is roughly just ϵn^2 more than $e(T_{r-1}(n))$.
- This is possible if G has $d(x) \geq n(1 - \frac{1}{r-1} + \epsilon)$ for every vertex $x \in V(G)$, under the same conditions. (See Lemma 9.11 in [PA11]).
- Clearly, $\epsilon < \frac{1}{r-1}$.
- The core idea is that pumping in so many edges into G would inflate the degrees of a bunch of vertices to cross the bound on vertex degrees required in Lemma 9.11, so that an induced subgraph of G has a K_r^t.

Erdos-Stone theorem (cont.)

- The harder part to show is the inductive proof of Lemma 9.11 as follows.
- The induction is on r.
- As we already know, the claim holds for $r = 2$, by the use of Corollary 9.7 to Lemma 9.6 and the Kovari, Sos, Turan result in Theorem 9.9 in [PA11], where ϵn^2 edges are sufficient.
- So, we can assume the claim holds for some $r \geq 2$ and using this claim we show it holds for $r + 1$ as well.
- Set $T = \lceil \frac{t}{\epsilon} \rceil$.
- If $n > n_1(r, T, \epsilon)$, then by the inductive hypothesis, $K_r^T \subseteq G$.
- We have to find vertices in $V(G)$ outside the K_r^T that are adjacent to at least t vertices in each partite of the K_r^T.
- Calling such vertices as *regular*, we need a sufficient number of them, say R, so that we can form a K_{r+1}^t.

Erdos-Stone theorem (cont.)

- We can lower bound R by $n^{\frac{\epsilon(r-1)}{1-\epsilon}} - rT$, by estimating the number m of missing or absent edges in G between the K_r^T and the rest of G, by using an easy upper bound on m based on vertex degree lower bounds in the premise of Lemma 9.11, and a non-trivial lower bound for m based on how the non-regular vertices miss out the adjacency of at least $T - t$ vertices in at least one partite of the K_r^T.
- Note that by choosing a sufficiently large n we can ensure the required number R of regular vertices necessary to build the K_{r+1}^t, as we state below.
- Let C_i, $i \leq i \leq \binom{T}{t}^r$, denote the ith combination of t-subsets from the r-partite sets V_j, $1 \leq j \leq r$, where each V_j has T vertices.
- Let $w_{i,k}$, $1 \leq k \leq t-1$ be a regular vertex adjacent to all vertices in C_i, $1 \leq i \leq \binom{T}{t}^r$.

Erdos-Stone theorem (cont.)

- So, adding just one more regular vertex x would create a K^t_{r+1}, because being regular, the vertex x must be adjacent to all vertices of some C_i, $1 \leq i \leq \binom{T}{t}^r$.
- So, just over $\binom{T}{t}^r(t-1)$ regular vertices suffice.
- Now using Lemma 9.11, the main Erdos-Stone theorem is proved in Theorem 9.10 of [PA11].
- For any $\epsilon > 0$, and we have at least $\frac{n^2}{2}(1 - \frac{1}{r-1} + \epsilon)$ edges in G, just $\frac{1}{2}\epsilon n^2$ more edges than $T_{r-1}(n)$.
- We will discriminate with respect to the vertex degree $d(x_i)$ being at least $(n-i)(1 - \frac{1}{r-1} + \frac{\epsilon}{2})$ or strictly lesser than this amount, respectively, to retain or drop the vertex x_i in Step i, $i \geq 0$, to set G_{i+1} as $G_i - x_i$.

Erdos-Stone theorem (cont.)

- When we get stuck at G_i with no more deletions possible, we are done with $|V(G_i)| \leq n(r, t, \frac{\epsilon}{2})$, whereby the Lemma 9.11 holds, assuring $K_r^t \subseteq G_i \subseteq G$, as sought.
- Setting a loose estimate of an upper bound on $|E(G)|$ based on the high vertex degrees of deleted vertices and the upper bound $\binom{n-i}{2}$ on the size of G_i, and the lower bound for $|E(G)|$ in the premise of the Theorem 9.10, we can derive a lower bound for $|V(G_i)|$ in terms of n, r and ϵ as being an estimate for the function $n(r, t, \frac{\epsilon}{2})$.
- Now as an application of the Erdos-Stone theorem, show that given a nonempty graph H with chromatic number $\chi(H)$, an H-free graph can have no more that $\frac{n^2}{2}(1 - \frac{1}{\chi(H)-1}) + o(n^2)$ edges.

The problem of K. Zarenkiewicz

- Kovari, Sos and Turan in 1954 solved the problem of bounding the number of edges in a bipartite graph $G_{m,n}(V_1 \cup V_2, E)$ so that another bipartite subgraph $K_{r,s}$ is forbidden.
- We wish to determine the maximum number $|E|$ of edges permissible so that a $K_{r,s}$ does not appear as a subgraph in a bipartite graph $G_{m,n}(V_1 \cup V_2, E)$.
- For all $x \in V_2$, (W, x) pairs must be at most $(s-1)\binom{m}{r}$ which must cap $\binom{d(x)}{r}$ summed over all $x \in V_2$, where W is a subset of V_1 of r vertices connected to the same $x \in V_2$.
- This is necessary because no r-tuple from V_1 should connect to more than $s-1$ vertices in V_2, whereas we do have as many as $\binom{d(x)}{r}$ r-tuples from V_1 connecting to each $x \in V_2$.
- So, essentially, we must pack all the available r-tuples we have from the graph into at most $(s-1)\binom{m}{r}$ r-tuples.

The problem of K. Zarenkiewicz (cont.)

- This forbids $K_{r,s}$ and breaching this packing restriction would make a $K_{r,s}$ appear.
- For a solution see Theorem 9.5, Combinatorial Geometry by Pach and Agarwal, Wiley Interscience Series in Discrete Mathematics and Optimization.
- The upper bound sought is $c_{r,s}(mn^{1-\frac{1}{r}} + n)$, where the constant $c_{r,s}$ depends only on r and s.
- Jensen's (secant) inequality comes in handy. The generalization to general graphs is easy and similar, as in Exercise 9.17 in [PA11].
- Do exercises 5.2.23, 5.2.25 and 5.2.26 from [Wes00].
- Indeed Exercise 9.17 in [PA11] helps establish Corollary 9.7 in [PA11] as G being $K_{r,s}$-free in Corollary 9.7 means even bipartite subgraph H in Exercise 9.17 is $K_{r,s}$-free.

The problem of K. Zarenkiewicz (cont.)

- Note that the case of a general graph being $K_{r,r}$-free requires no notion of orientation but only that any r-tuple in the vertex set $V(G)$ be contained in the neighbourhood of at most $r-1$ vertices.
- A vertex v with degree $d(v)$ has $\binom{d(v)}{r}$ r-tuples in its neighbourhood.
- The total number of r-tuples is $\sum_{v \in V(G)} \binom{d(v)}{r}$, which must not exceed $\binom{|V(G)|}{r}(r-1)$.
- Now using $\binom{x}{r}$ as a convex function $f(x)$ for $x \geq r-1$ and using Jensen's inequality show that $E(G)$ must be bounded by $C.n^{2-\frac{1}{r}}$ so that G has no $K_{r,r}$.

Tutte's theorem

- The generalization of Hall's theorem to general graphs, the result of Tutte, can be proved by using Hall's theorem (see Problem 3.3.13 in [Wes00]).
- It is easy to see that the necessity of Tutte's condition, for a simple graph $G(V, E)$, whereby $o(G - S) \leq |S|$, holds for every subset S of vertices.
- This is so as every odd connected component of $G - S$ would require to reserve at least one vertex in $S \subseteq V$ for the perfect matching.
- So, at least one edge of the perfect matching must connect each odd component of $G - S$ to S.
- So, if J is the set of all such edges over all odd components then $o(G - S) \leq |J| \leq |S|$.
- Since $|J|$ can be large, S will be larger, accomodating parallel edges of the matching to land up in S.

Tutte's theorem (cont.)

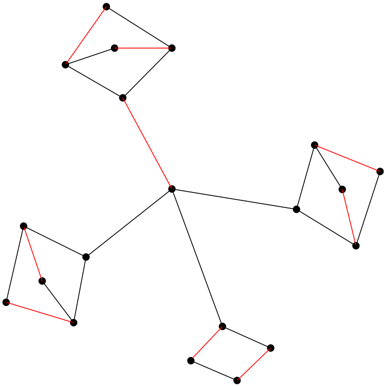


Figure: A graph with maximum matching of size 9.

- This also applies to even components but we do not mention anything about even components here !

Tutte's theorem (cont.)

- If G satisfies Tutte's condition and G has components, then furthermore, each component must also be even.
- If G has two vertices then satisfying Tutte's condition would enforce $G = K_2$, which is itself a perfect matching.
- For establishing sufficiency, first we note that if S is the empty set then Tutte's condition enforces that G must be even.
- Consider a simple graph G', constructed by adding edges to G, so that G' has no 1-factor but joining any pair of non-adjacent vertices by an edge in G' results in introducing a 1-factor.
- We show that such a graph G', has a special *dense* structure.
- If S is a bad set of G' then by its edge-maximality and Tutte's condition violation as per Tutte's theorem, all the components of $G' \setminus S$ are complete and every vertex $s \in S$ is adjacent to all the vertices of $G' \setminus \{s\}$, and S is a clique in G'.

The maximal graph idea for proving Tutte's theorem

- Consider the set V_1 of all vertices in G' that forms a clique and each vertex in V_1 connects to all vertices in $V \setminus V_1$. In other words, V_1 is the set of all vertices of degree $|V| - 1$ in G'.
- It can be shown that vertices outside V_1 span disjoint complete graphs, and each vertex $v \in V_1$ is adjacent to all vertices in $V \setminus \{v\}$. Such a set V_1 actually exists as we shall argue shortly.
- Using this property (established later below), we first prove Tutte's theorem as follows.
- We work on G' to establish that the negation of Tutte's condition holds for G'.
- Finally we show that there is an $X \subseteq V$ with $o(G - X) > |X|$ for the original connected simple graph G that has no perfect matching.
- So, let V_1 be the set of vertices where G' has the special structure with $G' - V_1$ consisting of complete subgraphs $G_1, G_2, \ldots, G_k$.

The maximal graph idea for proving Tutte's theorem (cont.)

- We need to show that there are at least $|V_1| + 1$ odd complete subgraphs amongst $G_1, G_2, ..., G_k$.
- For the sake of contradiction, assume that this is not the case.
- Since we assume thus for the sake of contradiction, we can now construct a perfect matching, contrary to the assumption that G' has no perfect matching.
- We can choose a 1-factor in each even G_i because each such component is a complete subgraph of G', and a maximum matching in for each odd G_i (each such matching missing one vertex) that includes a distinct vertex in V_1.
- So, we choose independent edges matching one vertex from each odd G_i, with some distinct vertex in V_1.

The maximal graph idea for proving Tutte's theorem (cont.)

- Now $|V|$ must be even (as we mentioned above) for G to obey Tutte's condition by choosing $X = \phi$. So, G' too is even.
- So, $|V|$ being even and with independent edges joining V_1 to odd G_i, we must have only an even number of vertices left in V_1 to match up arbitrarily, V_1 being a complete subgraph of G'.
- This holds always irrespective of whether there are an odd or an even number of odd G_i's.
- Thus we get a 1-factor, a contradiction.
- So, now we conclude that there are at least $|V_1| + 1$ odd components of $G' - V_1$.
- Finally, dropping added edges from G' to get G we may only increase the number of connected components and also the number of odd connected components, thereby satisfying $o(G - V_1) > |V_1|$.

Applications of Tutte's theorem

- Let $G = (V, E)$ be a connected 3-regular graph with no more than two cut-edges.
- Since the sum of degrees $3|V(G)|$ must be even, G must have an even number of vertices.
- Since we have at most two cut edges in G, we must have either zero, or one or two cut edges in G.
- The case of no cut edges is done in a simpler previous exercise.
- We know that for an arbitrary non-empty subset $S \subseteq V$ that there are (i) a edges going from S to the odd components of $G - S$, (ii) an even number b of edges going from S to the even components of $G - S$, and (iii) c edges connecting vertices within S.
- So, $a + b + 2c = 3|S|$. Let $o(G - S)$ be l.

Applications of Tutte's theorem (cont.)

- The induced subgraph of the odd component G_i has even sum of degrees $3|G_i| - m_i$, where m_i is the odd number of edges connecting this component to S.
- So, $o(G - S) = l$ is odd if and only if a is odd.
- If the cut edges are inside components of S then we can use the same argument as that given in a previous problem.
- So, we consider the case where the cut edge(s) are between a component and S.
- If there is only one cut edge then $3(l - 2) + 1 \leq a$, otherwise $3(l - 2) + 2 \leq a$.
- In either case, as l and a have identical parity, we have $3l - 4 \leq a$.
- So, we have $3l - 4 \leq a \leq a + b + 2c = 3|S|$.
- As $3|S|$ is divisible by 3 while $3l - 4$ is not, either $3l - 4 < a$ or $b + 2c > 0$.

Applications of Tutte's theorem (cont.)

- If $3l - 4 < a$, then actually $3l - 2 \leq a$, since both a and l are odd or both are even.
- So, $3l - 2 \leq 3|S|$, or $l \leq |S| + 2/3$.
- Otherwise, $3l - 4 = a$ and $b + 2c > 0$, so actually $b + 2c \geq 2$, since b is even.
- Why is b even?
- So, $a = 3l - 4 \leq 3l - 4 + b + 2c \leq 3|S|$, or $3l - 4 + 2 \leq 3|S|$, or again $l \leq |S| + 2/3$.
- A much shorter and simpler proof from Lovasz and Plummer's book "Matching Theory" is as follows.
- An elegant and shorter proof is from [LP86] for the Theorem 3.4.1, which was originally established by Petersen in his pioneering 1891 paper with a "somewhat tedious" proof, as mentioned in [LP86].

Applications of Tutte's theorem (cont.)

- In the shorter proof, we assume for the sake of contradiction that there is no perfect matching.
- So, we focus on the bad set S, which must have the same parity as $o(G-S)$ (Why?), as well as violate Tutte's condition, thereby satisfying $o(G-S) > |S|$.
- So, we must have $o(G-S) \geq |S| + 2$.
- So, the odd components of $G-S$ have a total of *at least* $3(o(G-S)-2)+2$ edges to/from S; at least 3 edges from all but two odd components, and at least 1 from two odd components.
- So, any odd component will require at least one (an odd number of) edge(s) to land in the bad set S, but at most two such components can have exactly one landing edge because there are at most two cut edges.

Applications of Tutte's theorem (cont.)

- The set of all edges of G, incident on the vertices of an odd component, is of odd cardinality, but edges that remain within the induced subgraph of the odd component are even in number.
- So, we have a total of at least $3|S| + 2$ edges from S connecting to these odd components, which is absurd.

Establishing Hall's theorem using Tutte's theorem

- This proof is based on https://math.stackexchange.com/questions/1803729/derive-halls-theorem-from-tuttes-theorem
- One way to visualize a perfect machine for a graph H is to look at an induced complete subgraph H' of H that has a perfect matching.
- Edges going out of $H - H'$ into H' can then also have a matching covering the vertices in $H - H'$, thus completing a perfect matching for H.
- We wish to now show how Hall's theorem follows from Tutte's theorem.
- Let G be a bipartite graph of n vertices with partites X and Y.
- Consider the graph H obtained from G by adding an extra vertex to the partite set Y if n is odd, and then adding edges between vertices in Y to make Y a complete induced subgraph.

Establishing Hall's theorem using Tutte's theorem (cont.)

- So, if $n = 2m$ then $H = G$ has n vertices, with Y having an odd (even) number k of vertices if X has an odd (even) number $n - k$ of vertices.
- If $n = 2m + 1$ then H has $2m + 2$ vertices with X having an odd (even) number k of vertices and Y has $2m + 2 - k$, an odd (even) number of vertices.
- It is easy to see that G has a matching of size $|X|$ if and only if H has a perfect matching.
- Assume that G has a matching of size $|X|$.
- Since G is bipartite, each of the $|X|$ edges of the matching has one endpoint in X and the other in Y, leaving the remaining vertices in $H(Y)$ to form a perfect matching (in the remaining complete subgraph) $H(Y)$, thus yielding a perfect matching for H.
- Suppose now that H has a perfect matching.

Establishing Hall's theorem using Tutte's theorem (cont.)

- Since $H[X]$ is independent, a perfect matching in H saturates X.
- What more we need to show now is that if G satisfies Hall's condition, then H satisfies Tutte's condition.
- Suppose that G satisfies Hall's condition. To verify that H satisfies Tutte's condition, we must show that $o(H - T) \leq |T|$ for every subset T of $V(H)$.
- Since $H[Y \cap T]$ is a clique, the odd components of $H - T$ are the vertices of X all of whose neighbors lie in T, possibly along with $Y - T$ (only if T is chosen so that $|Y - T|$ is odd).
- Let $S = \{x \in X, N(x) \subseteq Y \cap T\}$ (i.e. the vertices of X which become isolated upon the deletion of T from H).
- Since G satisfies Hall's condition, we have that $|S| \leq |T \cap Y| \leq |T|$, and thus $o(H - T) \leq |T| + 1$.

Establishing Hall's theorem using Tutte's theorem (cont.)

- However, since H is of even order, $o(H - T)$ and $|T|$ must have the same parity, and we obtain $o(H - T) \leq |T|$. Thus, H satisfies Tutte's condition.
- With the preceding steps/arguments, it is evident that Hall's theorem follows from Tutte's theorem.
- The necessity of Hall's condition is obvious (to have a matching which saturates X, any subset of X must have at least as many neighbors as elements in order to be completely matched).
- To see why sufficiency of Hall's theorem follows from Tutte's theorem, let H be the auxiliary graph considered throughout this proof. Since G satisfies Hall's condition (by assumption), H satisfies Tutte's condition as shown above.
- Since H satisfies Tutte's condition, it has a perfect matching (by Tutte's theorem).

Establishing Hall's theorem using Tutte's theorem (cont.)

- Finally, since H has a perfect matching, we may conclude that G has a matching of size $|X|$, as shown above in the beginning.

Hamiltonian circuits

- Study Theorems 4.2, 4.3 and 4.4 from [BM76]. Lemmas 4.4.1 and 4.4.2 in [BM76] are supporting results.
- Hamiltonian paths (or circuits) have all the vertices of the connected graph.
- Note that adding edges to the input graph G would not decrease vertex degrees.
- Addition of edges would also preserve Hamiltonian circuits.
- We add edges to G arbitrarily to create a graph G' which is non-Hamiltonian but on addition of any edge e, $G' + e$ is Hamiltonian.
- For the sake of contradiction, we will assume that G is not Hamiltonian.

Hamiltonian circuits (cont.)

- With these assumptions we will demonstrate the presence of two vertices in the input graph G whose degrees in G add up to a number strictly less than n, thereby contradicting the initial assumption that $\delta(G) \geq \frac{n}{2}$.
- This would mean that the graph G is Hamiltonian, given that the minimum vertex degree in G is at least half the number of nodes in the graph.
- This result is by Dirac. See [Wes00; Sur10].
- Since G' is assumed to be maximally non-Hamiltonian, we also know that G' is not complete.
- So, we take a pair uv where uv is not an edge of G'. However, $G' + uv$ is Hamiltonian.
- Any spanning cycle of G' must pass through uv in $G' + uv$, otherwise, we will get a spanning cycle in G'.

Hamiltonian circuits (cont.)

- So, omitting uv from $G' + uv$, we get a Hamiltonian path $u = v_1, v_2, ..., v_n = v$ in G'.
- We collect in a set S, every vertex v_i if $v_1 v_{i+1}$ is an edge.
- We also collect in a set T, all vertices v_i which are neighbours of $v = v_n$.
- Suppose a vertex v_i is common to S and T.
- Consider $v_1, v_2, ..., v_i, v_n, v_{n-1}, ..., v_{i+1}, v_1$.
- This is a Hamilton cycle in G' because $v_i v_n$ is an edge by virtue of $v_i \in T$, and $v_{i+1} v_1$ is an edge by virtue of $v_i \in S$.
- But this contradicts our assumption that G' is non-Hamiltonian.
- So, S and T must not share any element.
- Now $d(u) = |S|$ and $d(v) = |T|$.

Hamiltonian circuits (cont.)

- But S and T together have strictly less than n vertices (as neither S nor T can have v_n), a contradiction because each vertex degree in G' is at least $\frac{n}{2}$.
- This also contradicts the fact that the initial input graph G has minimum vertex degree at least $\frac{n}{2}$.
- Therefore, the assumption that G is not Hamiltonian is contradicted.
- The case of $n = 2$ is excluded here. So, we assume that the graph has at least three vertices.
- Note that we only used the sum of degrees of two vertices unconnected in G, which were at the two ends of the longest path, when we compared the sum of degrees with n.
- So, the minimum degree condition can be weakened/ generalized somewhat.

Hamiltonian circuits (cont.)

- So, we may now state a sufficient condition as follows. If $n > 3$ and the degree sum of any two non-adjacent vertices is at least n, then G contains a Hamilton circuit.
- This result is due to Ore. See [Wes00; Sur10].

Properties of the universal set of vertices of a maximally saturated graph with no perfect matching

- Let V_1 be the set of vertices connected to all other vertices by edges in G'.
- Let $V_2 = V \setminus V_1$.
- We want to show that if $a, b, c \in V_2$, and b is adjacent to both a and c, then a is adjacent to c.
- This means vertices in V_2 have adjacency as an equivalence relation, and therefore V_2 is partitioned into complete subgraphs.
- Suppose this is not the case.
- Since $b \in V_2$, there is a fourth point d which is not adjacent to b, as b would be in V_1 if it were adjacent to all vertices outside V_1.
- So, ac and bd are not in G'.
- By the maximality argument, $G' + ac$ has a 1-factor F_1 and $G' + bd$ has a 1 factor F_2.

Properties of the universal set of vertices of a maximally saturated graph with no perfect matching (cont.)

- Clearly, ac is a perfect matching edge in $G' + ac$, and bd is a perfect matching edge in $G' + bd$. Otherwise, G' would have had a perfect matching, which is not the case.
- Observe $F_1 \cup F_2$.
- This union has common edges and alternating circuits, ac on say C_1 and bd on say C_2.
- If C_1 and C_2 are distinct circuits then we get a 1-factor for G' (a contradiction), by replacing the edges of F_1 by the edges of F_2 inside the circuit C_1 (thereby removing the edge ac of F_1 from $G' + ac$), and yielding a (new) perfect matching in G'.

Properties of the universal set of vertices of a maximally saturated graph with no perfect matching (cont.)

- Note that bd was not there in G', and the F_2 edge bd in C_2 is not in C_1. So, the above switching will give a perfect matching in G', as ac is dropped in C_1, and edges of F_2 in C_1 now included in the new matching are all in G'.
- Thus F_1, a perfect matching for $G' + ac$ was modified to generate a new perfect matching for G' of the same size, a contradiction.
- The only other case is if $C_1 = C_2$.
- In the cycle $C_1 = C_2$ let us start from b through bd till we eventually hit ac, without loss of generality reach a before we hit c.
- This traversal $b, d, ..., a, c$ is a path from b to a, starting at F_2 edge bd, and ending at a in another F_2 edge, because ac is in $C_1 = C_2$ in F_1.

Properties of the universal set of vertices of a maximally saturated graph with no perfect matching (cont.)

- So, this (b, a) path P, plus the $\{a, b\}$ edge forms the cycle $K + ab$, with alternating edges of F_2.
- Replace edges of K in F_2 by edges of K not in F_2. This will give a new perfect matching in G' as bd of F_2 is dropped to get the new matching and bd was never in G'.
- We thus replace F_2 edges in this circuit $K + ab$ by an equal number of "other" alternating edges in $K + ab$ to get a 1-factor of G', whereby we drop $\{b, d\}$, which was not in G' anyway, and add $\{a, b\}$, which was in G'.
- Thus we get a new perfect matching for G', by modifying the perfect matching F_2 for $G' + bd$.
- However, G' does not have a 1-factor as we stated in our premise. So, we conclude that V_2 is an equivalence relation due to $\{a, c\}$ being in G'.

Perfect graphs

- Local conditions can lead to lower bounding the chromatic number though chromatic number can grow even if girth is high, as the graph grows.
- So, global conditions influence growth of chromatic number.
- However, in perfect graphs we have some checks.
- We define perfect graphs as those graphs G such that $\chi(H) = \omega(H)$ for every induced subgraph $H \subseteq G$.
- By this definition it is implied that induced subgraphs of perfect graphs are perfect.
- Since induced subgraphs of perfect graphs are perfect, it is natural to characterize perfect graphs using forbidden induced subgraphs.
- So, we can say that there is a set F of imperfect graphs such that any graph is perfect if and only if it has no induced subgraph *isomorphic* to any graph in F.

Perfect graphs (cont.)

- Here, F could be the set of all imperfect graphs but we would like to have a small set F.
- The most famous conjecture till 2005, due to Berge, from 1966, was that F is the set of all odd cycles of size least 5 and their complements.
- Such cycles and their complements are not perfect.
- So we can rephrase the conjecture – a graph G is perfect if and only if neither G nor its complement G' has an odd cycle of length 5 or more as an induced subgraph.
- This was the *strong perfect graph conjecture (SGPC)* which was settled as a theorem in 2005 [CRST06].
- Such graphs were known as Berge graphs, which we know now thus as exactly the perfect graphs.

Perfect graph theorem (PGT)

- The perfect graph theorem (PGT) states that a graph is perfect if and only if its complement is perfect.
- The SPGC clearly implies PGT.
- The PGT was proved by Lovasz in 1972. This proof involves two stages.

Generating perfect graphs by connecting two cliques

- Let G have two complete disjoint graphs and some edges between them. Then $\chi(G) = \omega(G)$.
- See Theorem 8.1 in [GGL95]; the complement graph of G is bipartite.

Generating perfect graphs by extension of a vertex

- Furthermore, in order to prove PGT, we require the following result (see Lemma 5.5.4 of [Die17]).
- The graph obtained by *expanding* a vertex of a perfect graph is also perfect.
- A vertex x in a graph G is expanded by adding a new vertex x' and connecting x' to x and all neighbours of x in G, thus obtaining the expanded graph G'.
- This result is established using induction on the number of vertices. Later PGT is also established using induction and using this vertex expansion result.
- Coming to expanding G at vertex x, introducing edge xx' by adding the new vertex x', we get graph G', where x' connects to all neighbours of x in G.
- We show that G' is perfect if G is so.

Generating perfect graphs by extension of a vertex (cont.)

- We use induction, with the basis case of expansion of K_1 to K_2, which are both perfect.
- Now G is perfect so for G' to be shown perfect we need only show $\chi(G') \leq \omega(G')$.
- This is so because every proper induced subgraph H of G' is either isomorphic to some induced subgraph of G (and therefore perfect with $\chi(H) \leq \omega(H)$), or created from a proper induced subgraph of G by the expansion of G.
- If it is the second case above then the induced subgraph H of G' must have x', and a proper induced subgraph K of G, with or without x.
- If x is not there then $H = K + \{x'\}$ is just like an isomorph of a proper induced subgraph $K + \{x\}$ of G where x' acts just like x.
- Otherwise we have the non-trivial case where both x and x' are in H !

Generating perfect graphs by extension of a vertex (cont.)

- In this case the subgraph H of G' is perfect, by the induction hypothesis and the expansion construction. Why?
- This is because we can use induction for showing that the extension of a proper induced perfect subgraph of G at the vertex x, yields a perfect graph H, even if it has both x and x' !
- So now we have shown that in all the possible cases for a proper subgraph H of G', H is indeed perfect and therefore has a $\omega(H)$ coloring.
- Therefore, now we only need to further show that $\chi(G') \leq \omega(G')$.
- Let $w = \omega(G)$, then $\omega(G')$ is either w or $w + 1$.
- The easier case is when the maximum clique size is $w + 1$.
- Then $\chi(G') \leq \chi(G) + 1 = w + 1 = \omega(G')$, because we may need just one more colour and G is perfect.

Generating perfect graphs by extension of a vertex (cont.)

- However if $\omega(G') = w$, then note that x is not in any K_w of G, as otherwise, together with x', that would yield a K_{w+1} in G', a contradiction to $\omega(G')$ being w.
- Observe that our definition of extension of G to G' at x by x' now helps us in using this trump card.
- Now G being perfect we color G with $\omega(G)$ colors.
- But x misses all K_w of G, though the color class X of x would not miss any K_w of G. Why?
- See the induced subgraph $H = G - (X \setminus \{x\})$, which misses the color class X but not x, and has $\omega(H) < w$.
- By the induction hypothesis (H being a proper induced subgraph of G, and thus being perfect), we can color H with $w - 1$ colors !

Generating perfect graphs by extension of a vertex (cont.)

- Now X is an independent set but observe from the expansion construction of x' that $X' = (X \setminus \{x\}) \cup \{x'\}$ is also an independent set as x and x' play similar connectivity roles, and this set X' is exactly all vertices in G' but not those in H by definition of H, X and x' !
- So the $(w-1)$-coloring of H can be extended to a w-coloring of G' by using only one additional color.

Generating a perfect graph by the extension at vertices with perfect graphs

- We will now establish a result by Lovasz of 1972 about graph extension where each vertex is extended by replacing it with some perfect graph.
- We will use the characterization of perfect graphs that uses the fact of an independent set in each and every induced subgraph meets all the maximum cliques in that induced subgraph.
- Imagine a vertex x_0 being replaced in G by some perfect graph $G(x_0)$ for constructing the extension G' of G at the vertex x_0.
- We can show that G' is perfect by only showing that $\chi(G') \leq \omega(G')$; structurally, all induced subgraphs of G' are similar to G' and therefore all the arguments applying to G' also apply similarly to these induced subsets.

Generating a perfect graph by the extension at vertices with perfect graphs (cont.)

- Let S be a color class in a $\chi(G)(=\omega(G))$-coloring of the perfect graph G containing x_0.
- Let S_1 be an independent set of G_{x_0}, that meets all maximum cliques of G_{x_0}.
- Why is $(S - x_0) \cup S_1$ an independent set in G'?
- Firstly x_0 is not in G' but S is an independent set, a color class of G's optimal coloring.
- And S_1 is a local independent set in G_{x_0}.
- Since S is the color class of x_0, no vertex in the replacement $G(x_0)$ is adjacent to any vertex of S.
- Suppose we take any maximum clique T of G'. Does T meet $(S - x_0)) \cup S_1$? Yes. Why?
- Hint: Both G and G_{x_0} are perfect.

Generating a perfect graph by the extension at vertices with perfect graphs (cont.)

- So now that we have seen how the extension G' of a perfect graph G is also perfect, we will proceed with proving PGT.

Proving PGT as in [Die17; GGL95]

- Induced subgraphs of perfect graphs are perfect.
- It suffices to show that given a perfect graph G, the complement graph $\bar{G}$ satisfies $\chi(\bar{G}) = \omega(\bar{G})$, because by the induction hypothesis, we know that every proper induced subgraph of $\bar{G}$ is also perfect because every induced subgraph of G is perfect.
- Let us consider the complete subgraph of K of G that meets all the maximum independent sets of G of size $\alpha(G) = \alpha$. Indeed there is such a subgraph K of G as we show below.
- Note that we have characterized perfect graphs as graphs G whose each induced subgraph H has an independent set in H that meets all maximum cliques of H.
- Observe that G and its induced subgraphs are perfect, and that the complements of the induced subgraphs of G are perfect, by the induction hypothesis.

Proving PGT as in [Die17; GGL95] (cont.)

- So, we have $\omega(\bar{G} - K) = \alpha(G - K) < \alpha(G) = \omega(\bar{G})$
- The strict inequality above follows because the independent set K in $\bar{G}$ meets all the maximum cliques of $\bar{G}$.
- In other words, each maximum independent set in G loses a vertex when K is dropped from G.
- Now let us start with a minimum proper vertex coloring with $\chi(\bar{G} - K)$ colors for $\bar{G} - K$ and add the independent set K of $\bar{G}$ to $\bar{G} - K$.
- Since K meets each maximum clique of $\bar{G}$, we may need a new color for the vertices in K for a minimum proper vertex coloring of $\bar{G}$ in addition to the $\chi(\bar{G} - K)$ colors required for $\bar{G} - K$.
- So, $\chi(\bar{G}) \leq \chi(\bar{G} - K) + 1 = \omega(\bar{G} - K) + 1 \leq \omega(\bar{G})$
- The equality above is by the induction hypothesis.

Proving PGT as in [Die17; GGL95] (cont.)

- Now all we need to do is show that there is a complete subgraph K in G that meets all the maximum independent sets in G.
- For the sake of contradiction we assume to the contrary that there is no such complete subgraph K.
- Then for every complete subgraph K of G we must have some maximum independent set A_K of G so that $K \cap A_K = \phi$.
- We will also require notation $\mathcal{K}$ for the set of all cliques $K_1, K_2, \ldots, K_t$ of G.
- For each vertex x of G, we count the number of $K \in \mathcal{K}$ such that x is a vertex in A_K, and call this count as $k(x)$; this will be the size of the clique that extends G at the vertex x in G.

$$k(x) = |\{K \in \mathcal{K} | x \in A_K\}|$$

Proving PGT as in [Die17; GGL95] (cont.)

- So, vertex x of G may vanish in the extension of G if $k(x) = 0$ but this does not affect perfectness of the extension.
- Now we determine $\omega(G')$ as the number of vertices of some maximum clique of G', where X is the corresponding maximum clique of G.
- We recall that the extension G' must be perfect because G is perfect and therefore $\chi(G') \leq \omega(G')$.
- We know that $\omega(G')$ must be the sum of all $k(x)$ such that $x \in X$, that is
$$w(G') = \sum_{x \in X} k(x)$$
which is the number of (x, K) pairs, where $x \in X$, and $K \in \mathcal{K}$ such that $x \in A_K$.
- This can be abbreviated as the sum over all $K \in \mathcal{K}$ of $|X \cap A_K|$, that is, $\sum_{K \in \mathcal{K}} |X \cap A_K|$.

Proving PGT as in [Die17; GGL95] (cont.)

- We also know that $|G'|$ must be the sum of $k(x)$ over all vertices x in G, which is the number of (x, K) pairs over all $K \in \mathcal{K}$ and $x \in V$), such that $x \in A_K$, abbreviated as the sum over all $K \in \mathcal{K}$ of $|V \cap A_K|$, which is clearly $|\mathcal{K}|.\alpha(G)$.
- Therefore, the easier part is to show that $|G'|$ is exactly $|\mathcal{K}|\alpha(\mathcal{G})$, thereby $\chi(G') \geq \frac{|G'|}{\alpha(G')} \geq \frac{G'}{\alpha(G)} = |\mathcal{K}|$.
- Why is $\alpha(G') \leq \alpha(G)$?
- The interesting part is showing that $\omega(G') = \sum_{x \in X} k(x) = \sum_{k \in \mathcal{K}} |X \cap A_K| \leq |\mathcal{K}| - 1$, thus deducing $\chi(G') \geq |\mathcal{K}| > |\mathcal{K}| - 1 \geq \omega(G')$.
- Now X being a complete subgraph cannot contribute more than one vertex to any A_K for $K \in \mathcal{K}$, where as $X \cap A_X = \phi$, whence $\omega(G') \leq |\mathcal{K}| - 1$.
- So we get $\omega(G') < \chi(G')$, contradicting perfectness of G'.

Chordal graphs are perfect

- See Propositions 5.5.1 and 5.5.2 in [Die17].

Characterizing perfect graphs using overlaps of all maximum cliques with an independent set

- Note that the very definition of perfect graphs (in terms of the equality of the chromatic number and the maximum cliques size of induced subgraphs) implies that proper induced subgraphs of perfect graphs are also perfect.
- Suppose G is a perfect graph and G' is a proper induced subgraph of G.
- So, G' being perfect as well, we have $\chi(G') = \omega(G')$.
- Therefore, any color class S of a $\chi(G')$ coloration must meet every maximum clique of G'.
- Otherwise, S misses some maximum clique say C of G' and therefore $|C| = \omega(G') < \chi(G')$, contradicting the perfectness of G'.
- So, G being perfect, we see that every induced subgraph G' of G has an independent set that meets every maximum clique of G'.

Characterizing perfect graphs using overlaps of all maximum cliques with an independent set (cont.)

- Conversely, suppose every proper induced subgraph G' of G has an independent set that meets all the maximum cliques of G'.
- We show that $\chi(G') \leq \omega(G')$ for the induced subgraphs G' by induction on the number of vertices in G'.
- Let S be an independent set in G' that meets every maximum clique in G'.
- Consider the induced subgraph $G' - S$.
- Clearly, any proper coloration of G' will not require at most one more color than $\chi(G' - S)$.
- So, $\chi(G') \leq \chi(G' - S) + 1$.
- Also, since S meets every maximum clique of G', adding S to $G' - S$ will give cliques of size $\omega(G') \geq \omega(G' - S) + 1$ in G'; S eats away at least one vertex from every maximum clique of G'.

Characterizing perfect graphs using overlaps of all maximum cliques with an independent set (cont.)

- Thus, we have $\omega(G') \geq \omega(G' - S) + 1 = \chi(G' - S) + 1 \geq \chi(G')$.
- The equality is due to the induction hypothesis.
- Now we will view the same result in a different way.
- For a perfect graph G, let H be any induced subgraph of G.
- Let A be any color class of a $\chi(H)(= \omega(H))$ coloration of H.
- So, the induced subgraph $H - A$ can be colored with strictly less than $\chi(H)$ colors, since a color class is fully deleted.
- Also, $H - A$ is perfect. So, $\omega(H - A) = \chi(H - A) < \chi(H) = \omega(H)$.
- Conversely, we show that if G is such that for every induced subgraph H of G, there exists an independent set A in H such that $\omega(H - A) < \omega(H)$, then G is perfect.
- If H is a proper induced subgraph of G then by the induction hypotheses H is perfect as H satisfies the premise of the converse.

Characterizing perfect graphs using overlaps of all maximum cliques with an independent set (cont.)

- So, we only need to consider G itself.
- So, assume that A is an independent set in G such that $\omega(G-A) < \omega(G)$.
- But $\chi(G-A) = \omega(G-A) < \omega(G)$, and also $\chi(G) \leq \chi(G-A) + 1 \leq \omega(G)$, since $G-A$ is perfect and G needs at most one more color than $G-A$ for a proper coloration.

Another characterization of perfect graphs by Lovasz (1972)

- A graph $G(V, E)$ is perfect if and only if $|V(H)| \leq \alpha(H)\omega(H)$ for ever induced subgraph H of G. Here, $V(H)$ is the vertex set of H.
- We will consider only the sufficiency part here as in Theorem 5.5.5 [Die17].
- The graph G may be assumed to be not perfect for the sake of contradiction, whereas the premise $|V(H)| \leq \alpha(H)\omega(H)$, holds for all induced subgraphs of G, including G itself.
- Also, by the induction hypothesis, each proper induced subgraph of G is perfect.

Another characterization of perfect graphs by Lovasz (1972) (cont.)

- (Equality 1:) For any non-empty independent set U of G, $G - U$ has the same values for chromatic number $\chi(G - U)$ and the maximum clique size $\omega(G - U)$, and this must be equal to say $\omega = \omega(G)$. Why? This naturally holds also for singleton vertex sets $U = \{u\}$, for $u \in V(G)$.
- Any $u \in V(G)$ may or may not be in a maximum clique K^{ω}, called K of G for brevity.
- (Fact 2:) If $u \notin K$ then K meets all the ω color classes of $G - u$. Why? Neither $G - u$ nor K have u, so K must have exactly one vertex from each color class of a proper vertex coloring of $G - u$ of ω colors.
- (Fact 3:) If $u \in K$, then K meets all the ω color classes of $G - u$ except one color class. Why? The vertex $u \in K$ is in one color class of a proper coloring of $G - u$, so K can cover one vertex from only $\omega - 1$ more color classes.

Another characterization of perfect graphs by Lovasz (1972) (cont.)

- Now starting from a maximum independent set $A_0 = \{u_1, ..., u_{\alpha(G)=\alpha}\}$ of G, we have $\omega(G)$ color classes $A_1, ..., A_\omega$ of a ω-coloring of $G - u_1$, as many color classes $A_{\omega+1}, ..., A_{2\omega}$ of a ω-coloring of $G - u_2$, and so on, a total of $\alpha(G)\omega(G)$ sets, A_1 through $A_{\alpha\omega}$.
- For each of the $\alpha\omega + 1$ such independent sets A_i, $i \in \{0, 1, 2, ..., \alpha\omega\}$, we know that $G - A_i$ has a K^ω, say K_i [by Equality 1].
- (Fact 4:) However, for each of the possibly multiple $K^{w\prime}s$, say K in G, we have $K \cap A_i = \phi$ for exactly one $i \in \{0, 1, 2, ..., \alpha\omega\}$.
- To see why Fact 4 holds, we use Facts 2 and 3.
- Observe that by Fact 2 above, if $K \cap A_0 = \phi$, that is, each of $u_i, 1 \leq i \leq \alpha$, misses K, then every color class of $G - u_i$ meets K, as follows.

Another characterization of perfect graphs by Lovasz (1972) (cont.)

- $K \cap A_{(i-1)\alpha+j} \neq \phi$, for all $j, 1 \leq j \leq \omega$.
- On the other hand, if $K \cap A_0 \neq \phi$, then $|K \cap A_0| = 1$, and $K \cap A_i = \phi$ for exactly one $i, 1 \leq i \leq \alpha\omega$.
- This happens as follows by using Facts 2 and 3.
- Let K meet A_0 at the unique vertex u_i. Apply Fact 3 to this unique vertex $u_i \in K \cap A_0$ (where only one of the color classes of $G - u_i$, that is, $A_{(i-1)\omega+j}$ would be missed by K), and apply Fact 2 to all the other vertices of A_0.
- Now define A as a matrix of row incidence vectors for A_i, and B as a matrix of column incidence vectors of K_j.
- Why is $AB = J$, where J is a $(\alpha\omega + 1)X(\alpha\omega + 1)$ matrix of all ones except the diagonal, which is all zeros? Hint: See how K_i is defined with respect to A_i.

Another characterization of perfect graphs by Lovasz (1972) (cont.)

- Are AB and thus A (B) non-singular?
- Why is the rank of A $\alpha\omega + 1$?
- Why is $|V(G)| \geq \alpha\omega + 1$? This contradicts the premise ($|V(G)| \leq \alpha\omega$), and therefore G must be perfect.

Bipartite graphs and their complements

- Are bipartite graphs perfect?
- Are complements of bipartite graphs also perfect?
- The Konig-Egervary theorem along with Gallai's theorem imply
- (i) $\beta(G) = \alpha'(G)$ (Konig-Egervary theorem)$=$,
- (ii) $n - \alpha(G)$ (independent sets and vertex covers in G are complements of each other)$=$,
- (iii) $n - \omega(G')$ (independent sets in G are cliques in the complement graph G')$=$,
- (iv) $n - \beta'(G)$(Gallai's theorem)$=$,
- (v) $n - \chi(G')$ for a bipartite graph G and its complement G', provided we show that,
- (vi) $\beta'(G) = \chi(G')$.
- Also, we have $\alpha(G) = \beta'(G)$ due to the Konig-Egervary theorem and Gallai's theorem.

Bipartite graphs and their complements (cont.)

- Is there a $\alpha(G) = \beta'(G) = \omega(G')$ vertex coloration for G', the complement of the bipartite graph G ?
- This above question would show $\chi(G') = \omega(G')$ for the complement G' of a bipartite graph G.
- Take this as a homework problem.
- Firstly, $V(G)$ can be covered by $\alpha(G) = \beta'(G)$ edges from the minimum edge cover of G, assuming G has no isolated points.
- The edge cover (clique cover) in G is made of as many stars as $\alpha'(G) = \beta(G)$.
- The minimum clique cover (edge cover) of size $\alpha(G) = \beta'(G) = \omega(G')$ in G corresponds to the minimum cover with independent sets in G', with as many colors required for a $\chi(G') = \alpha(G) = \beta'(G) = \omega(G')$ coloration.

Bipartite graphs and their complements (cont.)

- Incidentally, the centres the stars of the edge cover in G constitute a minimum vertex cover in G and single edges from the stars make a maximum matching in G.

Coloring

- We observe that edge coloring chromatic number $\chi'(G)$ of a graph G is the same as the vertex coloring chromatic number $\chi(L(G))$ of the line graph $L(G)$ of G.
- The edges $e = \{u, v\}$ of G incident at a vertex v of G form a clique C_v in $L(G)$ where each edge of e of G is a vertex v_e in $L(G)$ of the clique C_v.
- A trivial lower bound for $\chi'(G)$ is $\Delta(G)$, with equality attained (see Exercise 22 of Chapter 3 in [Bol98]) for bipartite graphs.
- Another lower bound is $\lceil \frac{e(G)}{\beta(G)} \rceil$.
- We note that a complete graph G requires to have $\binom{\chi(G)}{2}$ edges.
- It is also the case that any graph G has at least $\binom{\chi(G)}{2}$ edges.
- Suppose we construct a graph G vertex by vertex by adding edges incident on the new vertex at each step of vertex inclusion, where we call the current graph G' at each step.

Coloring (cont.)

- So, there will be steps of vertex inclusion when the chromatic number $\chi(G')$ is incremented.
- In these steps the new vertex must have been adjacent to vertices of as many as $\chi(G')$ different colors in any proper coloration of G'.
- So, the number of edges should increase by at least $\chi(G')$ to at least $\chi(G') + \binom{\chi(G')}{2} = \binom{\chi(G')+1}{2}$ in the new graph.
- A better way of looking at this is to visualize the $\chi(G)$ color classes, each of which is an independent set, and unless there are edges between two color classes, we can always merge the two into one class.
- Also, the sum of the vertex coloring numbers of a graph and its complement is no more than $n+1$, where n is the number of vertices. (Exercise 5 of Chapter 5 in [Bol98])

Coloring (cont.)

- This problem can be solved as in [Lov93] using a result of R. P. Gupta, in: Theory of Graphs. Proc. Int. Coll. Rome, Gordon and Breach, 1969.
- It is easy to see that we will need at least as many as $\frac{n(G)}{\alpha(G)}$ colors for proper vertex coloring.
- So, in any induced subgraph H of a perfect graph G, we have $\chi(H) = \omega(H) \geq \frac{n(H)}{\alpha(H)}$.
- This is also a sufficient condition for perfectness of G, and this also implies the perfect graph theorem PGT which says that a graph is perfect if and only if its complement graph is perfect.
- We may need more colors if there are vertices of high degree.
- However, it is easy to show that $\chi(G)$ will not exceed $\Delta(G) + 1$ in a greedy proper coloring.

Coloring (cont.)

- For the complete graph and the odd cycle, this is the best we can do as K_n has maximum degree $n-1$ and C_{2k+1} has maximum degree 2.
- Szekeres and Wilf (1968) gave an upper bound for chromatic number as follows. For any graph G, $\chi(G) \leq 1 + \max \delta(G')$, where the maximum is taken over all induced subgraphs G' of G.
- So, for K_n, we have $\chi(K_n) = (n-1) + 1$. For C_{2k+1} the chromatic number is 2+1.
- So, much for these extreme cases of regular graphs.
- Now consider a non-regular graph $K_4 - \{e\}$. where e is any edge in K_4.
- This graph has 4 vertices, 5 edges, and 15 non-empty vertex subset induced subgraphs, of which the four singleton vertex subset subgraphs have minimum vertex degree 0, five 2-vertex subgraphs have minimum degree 1, one 2-vertex subgraph has 0, two 3-vertex subgraphs 2, two 3-vertex subgraph 1, and one 4-vertex graph 2.

Coloring (cont.)

- So, we see that $\chi(K_4 - \{e\}) = 1 + 2$, satisfying the upper bound inequality for $\chi(G)$ tightly.
- Let $k = \max \delta(G')$, where G' is any induced subgraph of G.
- So, the graph G must have a vertex of degree at most k.
- This is subtle and deep, and would require some thought. So, we give a few examples.
- In the case $G = K_n$, $k = n - 1$ and this is also the degree of a vertex in the minimal subgraph of K_n with chromatic number $\chi(K_n) = n$.
- For $G = K_4 - \{e\}$, $k = 2$, and this is also the degree of a vertex in the minimal subgraph of $K_n - \{e\}$ with chromatic number $\chi(K_n - \{e\}) = 3$.
- Another example is the *wheel graph* W_6, which has a 5-cycle and a 5-star, 10 edges, 6 vertices, girth 3, maximum clique size 3, chromatic number 4, and the minimum degree of all its proper induced subgraphs is at most 2, whereas minimum degree of W_6 is 3.

Coloring (cont.)

- So, $\chi(G) \leq 1 + \max \delta(G') = 1 + 3 = 4$, over all induced subgraphs G' of $G = W_6$.
- Note that W_6 has no K_4 but has chromatic number 4.
- However, like W_6, W_7 too has all its proper induced subgraphs of minimum degree at most 2, whereas W_7 has minimum degree 3, with chromatic number $3 \leq 1 + 3$, loosely satisfying the inequality.
- So, W_7 fits the inequality $\chi(G) \leq 1 + \max \delta(G')$, over all induced subgraphs G' of $G = W_7$, loosely.
- Now suppose in an arbitrary connected graph G, let x_n be a vertex that has degree no more than the maximum of the minimum vertex degree, $k = \max \delta(G')$, over all induced subgraphs G' of G.
- Observe that by the definitions, it holds that G has such a vertex x_n.
- Set $H_{n-1} = G - \{x_n\}$.

Coloring (cont.)

- Further observe that H_{n-1} too has a vertex of degree at most k. Let x_{n-1} be such a vertex.
- Set $H_{n-2} = H_{n-1} - \{x_{n-1}\} = G - \{x_n, x_{n-1}\}$.
- In this way we enumerate all the vertices of G.
- Observe that greedily coloring vertices in the order of this ingenious sequence $x_1, x_2, ..., x_n$, each x_j connected by an edge to at most k vertices preceding it in the sequence.
- This result is deep and can be exploited in the case of connected graphs that are not Δ-regular where Δ is the maximum vertex degree.
- Observe the subtle property that in such cases, $\max \delta(H)$ over all induced subgraphs H of G is at most $\Delta - 1$, making $\chi(G) \leq \Delta$.
- So, we need to consider now only Δ-regular connected graphs G for showing that $\chi(G) \leq \Delta$, for establishing Brooks' theorem.
- This is shown in Theorem 3, Chapter 5 in [Bol98] as follows.

Coloring (cont.)

- A graph can be disconnected or may have a cut vertex or, contain a complete subgraph whose vertex set disconnects the graph, as in chordal graphs.
- In such cases we can colour each part separately and then combine these colourings to produce a colouring of the whole graph.
- Therefore, we may assume without loss of generality that G is 2-connected and Δ-regular.
- Here, we drop the case of $\Delta = 2$ because a connected 2-regular 3-chromatic graph is an odd cycle.
- So, we assume $\Delta \geq 3$.
- If G is 3-connected, let x_n be any vertex of G and let x_l, x_2 be two nonadjacent vertices in $G - \{x_n\}$ in the neighbourhood of x_n in G.
- Such vertices exist since G is regular and not complete.

Coloring (cont.)

- If G is not 3-connected, let x_n be a vertex for which $G - \{x_n\}$ is separable, and thus has at least two blocks.
- Since G is 2-connected, each endblock of $G - \{x_n\}$ has a vertex adjacent to x_n.
- Let x_1, x_2 be such vertices belonging to different endblocks.
- In either case, we have found vertices x_l, x_2, x_n such that $G - \{x_1, x_2\}$ is connected, $x_1 x_2$ is not an edge of G, x_n has edges to x_1 and x_2.
- Let $x_{n-1} \in V - \{x_l, x_2, x_n\}$ be a neighbour of x_n, let x_{n-2} be a neighbour of x_n or x_{n-1}, etc.
- Then the order $x_l, x_2, x_3, ..., x_n$ is such that each vertex other than x_n is adjacent to at least one vertex following it.
- Thus the greedy algorithm will use at most $\Delta(G)$ colours, where x_1 and x_2 get the same colour.

Coloring (cont.)

- Here, x_n is the only vertex with Δ neighbours preceding it, x_n being adjacent to both x_1 and x_2, whereas while coloring each of the other $n-1$ vertices greedily we have at most $\Delta - 1$ neighbours to consider.
- We now present an alternative proof of the Szekeres-Wilf result as in [Har69].
- Let $\chi(G) = k$ and let H be a minimal induced subgraph of G with $\chi(H) = k$.
- Let v be any vertex of H.
- Then $\chi(H - v) = k - 1$.
- So, $d(v) \geq k - 1$ and $\delta(H) \geq k - 1$.
- So, the maximum of $\delta(H')$ over all induced subgraphs H' of H is at least $k - 1$.
- It is now easy to see that $\max \delta(G')$ over all induced subgraphs G' of G is also at least $k - 1$.

Hypergraph theory

- Let us consider a set system or a hypergraph $H = (V(H), E(H))$.
- A subset $A \subseteq V(H)$ is called *shattered* if for every $B \subseteq A$, there exists an $E \in E(H)$ such that $E \cap A = B$.
- The Vapnik-Chervonenkis dimension $VC - dim(H)$ of H is defined as the cardinality of the largest shattered subset of $V(H)$.
- If $VC - dim(H)$ is d then we show that $|E(H)| \leq \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{d}$.
- This is trivial if $d = 0$ and $n \leq d$. Why?
- For $d = 0$ and $n < d$, it is vacuously true.
- For $n = d$, we have all possible subsets of $V(H)$ realized as edges in $E(H)$. See [PA11].
- We use strong induction on both d and n.
- We must drop a vertex to do induction on n; to do induction on n and d simultaneously, we must drop a vertex as well as edges which did not have the vertex.

Hypergraph theory (cont.)

- In the former case we just take the proper induced hypergraph by dropping a vertex $x \in V(H)$.
- In the latter case, we retain only those $E \in E(H)$ edges that do not have the deleted vertex x but $E \cup \{x\} \in E(H)$.
- Naturally, these two sets $E(H1)$ and $E(H2)$, are such that $|E(H1)| + |E(H2)| = |E(H)| \leq \Sigma_{i=0}^{d} \binom{n-1}{i} + \Sigma_{i=0}^{d-1} \binom{n-1}{i} = \Sigma_{0}^{d} \binom{n}{i}$.
- In the latter case the VC dimension too falls with the deletion of the vertex and edges.
- This because if $E(H2)$ shatters a set $A \subseteq V(H) - \{x\}$ then $E(H)$ shatters the set $A \cup \{x\}$.
- Suppose$B \subseteq A$ is such that $B = h \cap A$ for some $h \in E(H2)$, then $B \cup \{x\} = (h \cup \{x\}) \cap (A \cup \{x\})$.
- This means that the upper bound on the number of edges forces the VC dimension to rise as we add edges.

Hypergraph theory (cont.)

- The *vertex cover* number or *transversal* number $\tau(H)$ is naturally defined by extending the similar notions in graphs.
- The *packing* number or *matching* number $\nu(H)$ is the dual counterpart, whereby $\nu(H) \leq \tau(H)$.

Planarity

- See Proposition 6.1.2 in [Wes00].
- For any planar drawing without crossings for a graph, we can start drawing a spanning cycle first (if any), in a closed loop, and then draw internal and external chords for remaining edges judiciously, without crossings.
- We can show that Kuratowski's two graphs are not planar by showing that it is impossible to embed the graphs thus, by analysis of the various exhaustive cases.
- The easier part of Kuratowski's theorem is to show that the presence of homeomorphs of K_5 or $K_{3,3}$ as subgraphs would make a graph non-planar.
- We achive this by (i) showing that K_5 and $K_{3,3}$ are non-planar, and (ii) the presence of a homeomorph of a non-planar graph causes non-planarity.

Planarity (cont.)

- A graph G is a homeomorph of another graph H if G can be obtained by repeatedly adding degree-2 vertices w by deleting edge $\{u, v\}$, and adding edges $\{u, w\}$ and $\{w, v\}$.
- Note that H is planar if and only if its homeomorph G is planar.
- The necessary condition in Kuratowski's theorem is that homeomorphs of none of the two Kuratowski's graphs can appear as subgraphs in a planar graph.
- The tougher (sufficiency) part of Kuratwoski's theorem is to show that a graph is planar if it does not have subgraphs homeomorphic to the any of the two Kuratowski graphs.
- We can show that a connected simple planar graph with m edges, n vertices and girth g satisfies $m \leq \frac{g(n-2)}{g-2}$.
- The dual of a planar embedding of a planar graph is such that the sum of degrees of the faces in the planar embedding is $2m$, exactly the same as the sum of degrees of the vertices.

Planarity (cont.)

- The degree of a face is the number of its bounding edges.
- So, $2m \geq gf$ where f is the number of faces.
- Now use Euler's equation $n + f = m + 2$. For $K_{3,3}$, $m = 9$, $g = 4$ and $n = 6$, this inequality is violated.
- The *thickness* of G is the least integer k so that G has *planar partition* $[G_1, G_2, ..., G_k]$.
- A *planar partition* of G is a collection $\mathcal{G} = [G_1, G_2, ..., G_k]$ of edge-disjoint spanning subgraphs of G, whose union is G.
- We can derive a lower bound for the thickness $\theta(G)$ of G in terms of the number m of edges of G, the girth g of G, and the number of vertices n of G.
- *Blocks* of a graph must be planar for a graph to be planar. So, planarity of blocks is a necessity.
- Is this also sufficient? See Theorem 11.2 in [Har69].

Planarity (cont.)

- The following notations and definitions are from [Har69].
- *Cutpoints* are cut vertices whose removal increases the number of components.
- A *nonseparable graph* is nontrivial, connected and has no cutpoints.
- Note that complete graphs are nonseparable. A *block* is a maximal nonseparable graph. *Bridges* are cut edges.
- Reading exercises: Theorems 3.1 through 3.4 from [Har69] on cut vertices and cut edges.
- Work out Problems 3.12 and 3.13 from [Har69] on the number of blocks and cut vertices, respectively.
- Following [Par94], we proceed as follows restricting our attention to 3-connected graphs and blocks.
- Corollary 6.8 in [Par94] is the simpler part of necessity, stating that any graph containing a homeomorph of K_5 or $K_{\{3,3\}}$, is non-planar.

Planarity (cont.)

- The sufficiency can be established by just proving the converse for 3-connected graphs, using Lemma 6.2 of [Par94], which says that a non-planar graph G with the minimal number of edges that contains no subdivision of the two Kuratowski graphs, is simple and 3-connected.
- This Lemma 6.2 of [Par94] is stated equivalently in Lemma 6.2.7 of [Wes00] as"If G is a graph with fewest number of edges among all non-planar graphs without Kuratowski subgraphs, then G is 3-connected."
- We know that a minimal non-planar graph is a block.
- For the sake of contradiction for Lemma 6.2 of [Par94], suppose the non-planar given graph G, with a minimal number of vertices, has a 2-vertex cut $S = \{u, v\}$, and is thus not 3-connected.
- Let $G - S$ have G_1 and one connected component and G_2 as the union of the rest of the connected components.

Planarity (cont.)

- Let $H_1(H_2)$ be the induced subgraph with vertex set $V(G_1) \cup S$ $(V(G_2) \cup S)$, with both graphs added with an additional edge $e = uv$. What happens in the other case where uv is an edge of G?
- Also, we will now show that at least one of H_1 and H_2 is non-planar because G is non-planar, which is easy to see as in [Par94].
- For the sake of contradiction, suppose the H_1 is non-planar. What if instead H_2 is non-planar?
- Then, the non-planar graph H_1 not being a subgraph of G (as xy is not an edge of G), but H_1 being smaller than G in the number of edges, H_1 must have a subdivision K of one of the two Kuratowski graphs, by the minimality of G.
- However, K must necessarily have e because $K \subseteq G$ would contradict our assumption that G has no subdivision of any of the two Kuratowski graphs.

Planarity (cont.)

- Now, replace e of K by a u to v path in H_2 to get a homeomorph of K in G, a contradiction.
- So, G must have no 2-vertex cut and so G must be 3-connected.
- The case where xy is an edge of G can be very similarly argued.
- Now we can show that the sufficiency condition for planarity holds for 3-connected graphs and that would be enough to do. Why?
- So, let G be a connected non-planar graph which is 3-connected. We discuss the proof of Thomassen's result as stated in the proof of Theorem 6.2 in [Par94] after we state a few more elementary results.
- See Section 6.2 of [Wes00], Lemmas 6.2.7, 6.2.6, 6.2.5 and 6.2.4, in that order for a detailed top-down presentation of the main result about considering only 3-connected graphs, as in Lemma 6.2 of [Par94].
- Definition 6.2.3 for "Kuratowski subgraphs", "minimal non-planar graphs" in [Wes00], and Definition 5.2.19 in [Wes00] will be useful.

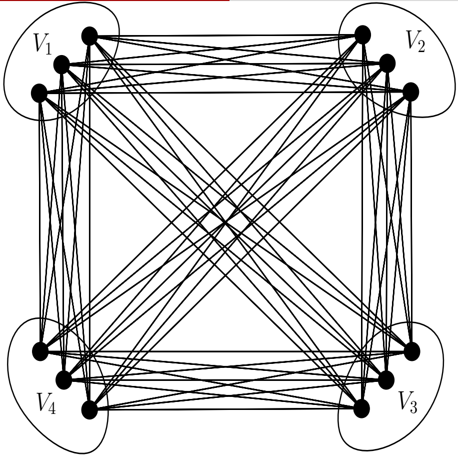


Figure: $T_4^3(12)$, the Turan graph of 12 vertices, 4-partite, with three vertices in each partite and thus also the K_4^3. This graph has multiple K_4's but is just one edge deficient from possessing a K_5.

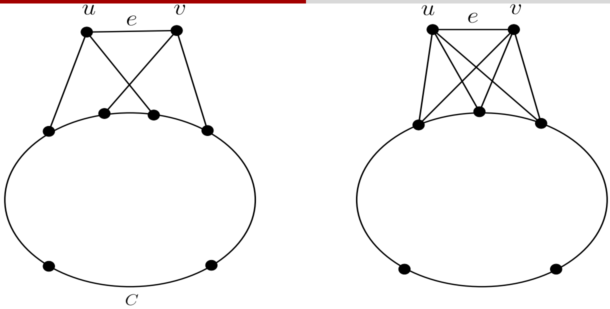


Figure: The non-planar cases of $K_{3,3}$ and K_5 respectively, appearing as illustrated in Figure 6.7(a) [Par94].

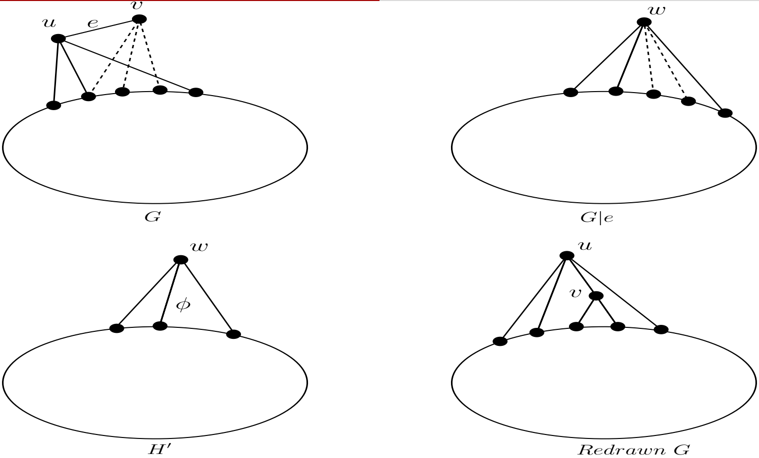


Figure: The transformations not violating planarity in the planar case, as illustrated in Figure 6.7(b) [Par94].

Another proof of Kuratowski's theorem

- Let G be a *minimal* non-planar graph with all vertices having degrees at least three. [By minimality we mean that each proper subgraph of G is planar.]
  1. We first how that G is 3-connected. (Part (a)). (Problem 5.37(a) [Lov93].)
  2. For the sake of contradiction, assume that G is not 3-connected. However, it is trivially 2-connected as the minimum degree is three in the connected graph G.
  3. So take any a two-vertex separator $S = \{x, y\}$ in G. Then, we can define two separated graphs as follows, which are both planar and then use their planar embeddings to get a planar embedding for G, a contradiction.
  4. Let G_1 and G_2 be such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{x, y\}$.
  5. Let P_1 (P_2) be an (x, y)-path in G_1 (G_2) and $H_1 = G_1 + P_2$ ($H_2 = G_2 + P_1$).
  6. Observe that H_1 and H_2 are both planar !

Another proof of Kuratowski's theorem (cont.)

7. Now deleting the two paths P_1 and P_2 between x and y, we get a planar embedding of G, a contradiction.
8. Now we further show that G has a cycle with a chord. (Problem 5.37(b) [Lov93]). (Part (b)).
   [Hint: Observe a longest path and that its first vertex has two more neighbours in the same longest path.]
9. Also, G must be isomorphic to a K_5 or a $K_{3,3}$. (Problem 5.37(c) [Lov93]). (Part (c)).
   [Hint: Use circuits and chords: Remove the chord xy of the circuit C that encloses the largest number of connected components of $G - C$. All such components must be inside C. Prove that the graph has only chords outside the circuit. Now consider the bridges inside the circuit and flap out those that are flapable outside C. Show that there now must be a bridge that forms a K_5 or a $K_{3,3}$ along with xy.]

Another proof of Kuratowski's theorem (cont.)

10. Show that a graph G that is non-planar must have a subdivision of a Kuratowski subgraph. (Problem 5.37(d) [Lov93]).
[Hint: A planar graph cannot have any Kuratowski subgraph. However, we must now show that a non-planar graph must have a subdivision of a Kuratowski subgraph. Use Parts (a), (b) and (c) above.]
[Solution sketch: If G is not planar then G contains a minimal non-planar graph G_0. If we get rid of the vertices of degree 2, we get another minimal non-planar graph, now with vertex degrees at least three. This graph is must be either of the two Kuratowski graphs. So, G_0 must be a subdivision of one of the two Kuratowski graphs.]

The Kuhn-Munkres theorem

- We start with the initial feasible labeling $l(x) = max_y w(x, y)$, of vertices in the complete bipartite graph $G(X, Y, E)$, with non-negative weight edges, where $y \in Y$ for each $x \in X$, and $l(y) = 0$, for all $y \in Y$.
- It is easy to see that this is a feasible labeling, that is, $l(x) + l(y) \geq w(x, y)$, for every edge $(x, y) \in E$.
- Moreover, each $x \in X$ is connected to at least one vertex $y \in Y$ where the equality holds (actually, to all the vertices in Y for which the maximum outgoing weight from x is assigned to $l(x)$).
- Therefore, the equality graph $G_l(X, Y, E_l)$ is not empty, to begin with.
- However, we do not know how many edges of G(X,Y,E) are there in E_l or whether G_l has a perfect matching.
- We can definitely compute the maximum (cardinality) matching in G_l, which may not be a perfect matching.

The Kuhn-Munkres theorem (cont.)

- We plan to compute a perfect matching in G_l.
- If G_l has no perfect matching, we may improve the labeling l of vertices to compute another feasible labeling l such that it now contains more "useful" edges.
- Some edges may be lost as we go from G_l to $G_{l'}$, and some edges may be added.
- We may then again compute the maximum matching in the new equality graph $G_{l'}$ and see if this matching is a perfect matching, because $G_{l'}$ would have an augmenting path, with respect to the maximum matching of G_l.
- Kuhn–Munkres Theorem: Let l be a feasible labeling of G. If M is a perfect matching in G_l, then M is a maximum matching in G.

The Kuhn-Munkres theorem (cont.)

- Let M be any matching in G. We then have $w(M) = \sum_{(x,y)\in M} w(x,y) \le \sum_{(x,y)\in M} [l(x) + l(y)] \le \sum_{x\in X} l(x) + \sum_{y\in Y} l(y) = \sum_{v\in V} l(v)$, by the feasibility requirement of vertex labels/weights.
- So, we have two equalities sandwiching two inequalities.
- The first inequality is because the l function is perhaps not yet optimized.
- The second inequality is because M may not be a perfect matching in G_l.
- If M is a perfect matching in G_l, it matches all vertices in V.
- Therefore, for a perfect matching M of G_l, we have $w(M) = \sum_{(x,y)\in M} w(x,y) = \sum_{(x,y)\in M} [l(x) + l(y)] = \sum_{x\in X} l(x) + \sum_{y\in Y} l(y) = \sum_{v\in V} l(v)$, because forcing the second inequality into an equality, forces also the first inequality to become an equality.

The Kuhn-Munkres theorem (cont.)

- Now consider forcing the first inequality $w(M) = \sum_{(x,y)\in M}[l(x) + l(y)]$. Then, we show that M is a perfect matching in G_l. This is intuitively so because the equality $w(x, y) = l(x) + l(y)$ will now hold for each edge (x, y) in M, making M a perfect matching in the equality graph G_l. This is due to the fact that all $w(x, y)$, $l(x)$ and $l(y)$ are non-negative. See Lemma 3.2.7 in [Wes00].
- A more rigorous argument is there in Lemma 13.2.2 in [Jun99].

The Hungarian algorithm

- Enhancing the equality graph requires looking for an edge that minimally violates the equality.
- So, we build the “alternating tree” that finally gives an augmentation, and this is carried on until we get a perfect matching in the equality graph.

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