CS60021: Scalable Data Mining

Subset Selection

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Submodular Subset Selection

Slides taken from IJCAI 2020 tutorial by
Rishabh Iyer and Ganesh Ramakrishnan

Combinatorial Subset Selection Problems

$V = \{$

$\}$

$$f : 2^V \to \mathbb{R}$$

$A = \{$

$\}$

Choose Subset $A \subseteq V$

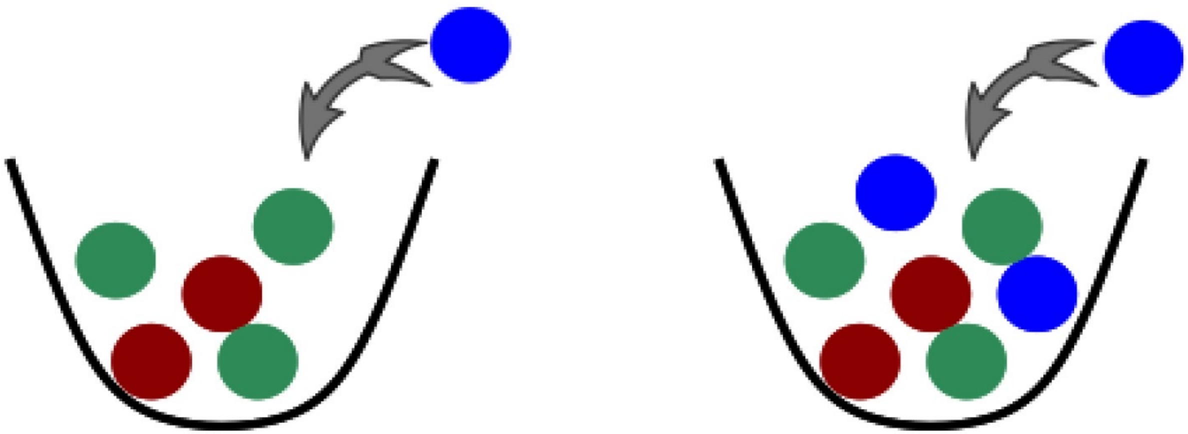
$f(A)$ is maximum

General Set function Optimization: very hard!

What if there is some special structure?

Submodular Functions

$$f(A \cup v) - f(A) \geq f(B \cup v) - f(B), \text{ if } A \subseteq B$$



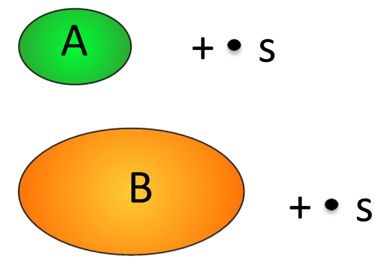
$f = \#$ of distinct colors of balls in the urn.

Negative of a Submodular Function is a Supermodular Function!

Equivalent Definitions of Submodularity

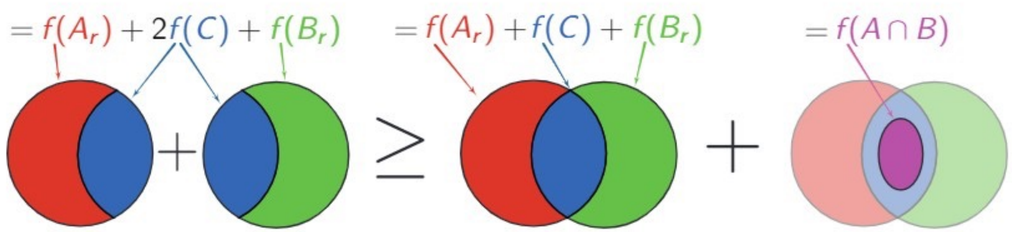
- **Diminishing gains**: for all $A, B \subseteq V$

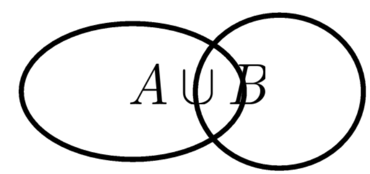
$$f(A \cup v) - f(A) \geq f(B \cup v) - f(B), \text{ if } A \subseteq B$$



- **Union-Intersection**: for all $A, B \subseteq V$

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$





Equivalent Definitions of Submodularity

Lemma: The above definitions for submodularity are equivalent.

Proof: We first assume that for all $A, B \subset S$, we have

$$f(A \cap B) + f(A \cup B) \leq f(A) + f(B).$$

Suppose that $A \subset B$, then for any $i \in S \setminus B$, we have that

$$\begin{aligned} f(A \cup \{i\}) + f(B) &\geq f(A \cup B \cup \{i\}) + f((A \cup \{i\}) \cap B) \\ &= f(B \cup \{i\}) + f(A), \end{aligned}$$

where the equality holds since $A \subset B$.

Equivalent Definitions of Submodularity

We now assume that

$$f(A \cup \{i\}) - f(A) \geq f(B \cup \{i\}) - f(B)$$

for each $A \subset B \subset S$ and $i \in S \setminus B$.

Consider any two sets A and B. If $A \setminus B = \emptyset$, then we have $A \subseteq B$, and thus

$$f(A \cap B) + f(A \cup B) = f(A) + f(B) \leq f(A) + f(B).$$

Otherwise, let $B \setminus A = \{v_1, v_2, \ldots, v_n\}$ and denote $X_i = \{v_1, v_2, \ldots, v_i\}$ and $X_0 = \emptyset$. Since $(A \cap B) \cup X_i \subset A \cup X_i$ We thus have

$$f((A \cap B) \cup X_i \cup \{v_{i+1}\}) - f((A \cap B) \cup X_i) \geq f((A \cup X_i) \cup \{v_{i+1}\}) - f((A \cup X_i),$$

that is

$$f((A \cap B) \cup X_{i+1}) - f((A \cap B) \cup X_i) \geq f(A \cup X_{i+1}) - f(A \cup X_i).$$

Summing from $i = 0$ to $n - 1$, and we yield

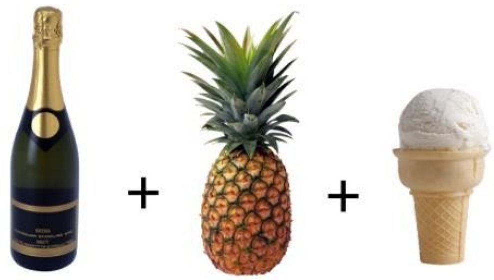
$$f((A \cap B) \cup X_n) - f(A \cap B) \geq f(A \cup X_n) - f(A).$$

Combined with $X_n = B \setminus A$, we have

$$f(A \cap B) + f(A \cup B) \leq f(A) + f(B).$$

Modular Functions

- each element e has a weight $w(e)$



$$F(S) = \sum_{e \in S} w(e)$$

$A \subset B$

$$F(A \cup e) - F(A) = w(e) \quad = \quad F(B \cup e) - F(B) = w(e)$$

Modular Functions are both submodular and supermodular!

Monotone Submodular Functions

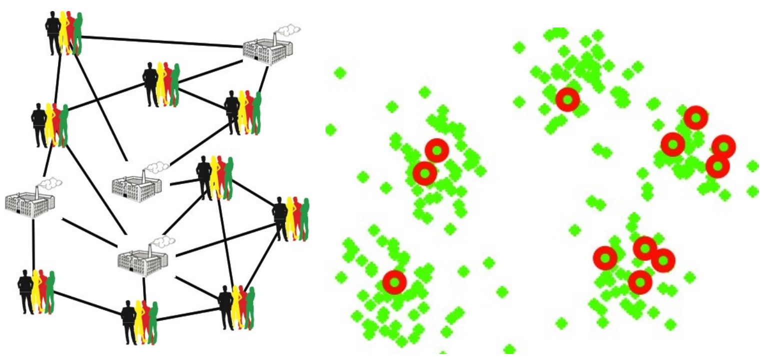
- A set function is called monotonic if
  $$A \subseteq B \subseteq V \Rightarrow F(A) \leq F(B)$$

- Examples:
  - Influence in social networks [Kempe et al KDD '03]
  - For discrete RVs, entropy $F(A) = H(X_A)$ is monotonic:
    Suppose $B = A \cup C$. Then
    $$F(B) = H(X_A, X_C) = H(X_A) + H(X_C \mid X_A) \geq H(X_A) = F(A)$$
  - Information gain: $F(A) = H(Y) - H(Y \mid X_A)$

Instantiations of Submodular Functions

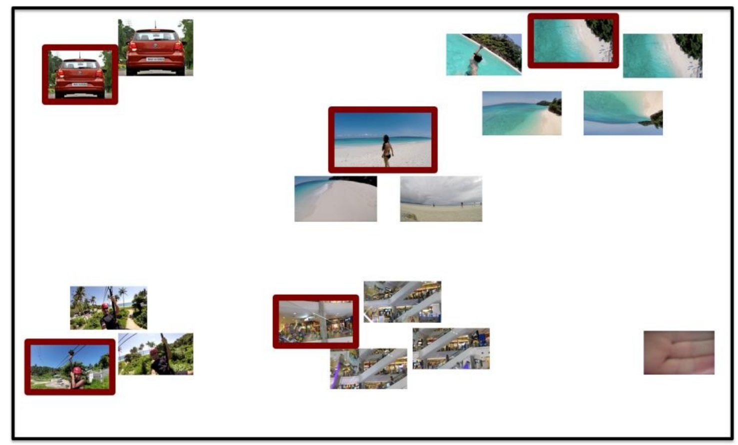
- **Representation Functions**
  - Facility Location Function (k-mediods clustering)
  - Graph Cut Family, Saturated Coverage
- **Diversity Functions**
  - Dispersion Functions (Min, Sum, Min-Sum)
  - Determinantal Point Processes
- **Coverage Functions**
  - Set Cover Function
  - Probabilistic Set Cover Function
  - Feature Based Functions
- **Importance Functions**
  - Modular Functions
- **Information Functions**
  - Mutual Information
  - Entropy
- **Discounted Cost Functions**
  - Clustered Concave over Modular Functions
  - Cooperative Costs and Saturations
- **Complexity Functions**
  - Bipartite Neighborhood Functions

Representation Functions



Facility Location	$\sum_{i \in V} \max_{k \in X} s_{ik}$
Saturated Coverage	$\sum_{i \in V} \min\{\sum_{j \in X} s_{ij}, \alpha_i\}$
Graph Cut	$\lambda \sum_{i \in V} \sum_{j \in X} s_{ij} - \sum_{i,j \in X} s_{ij}$

Similarity Kernel

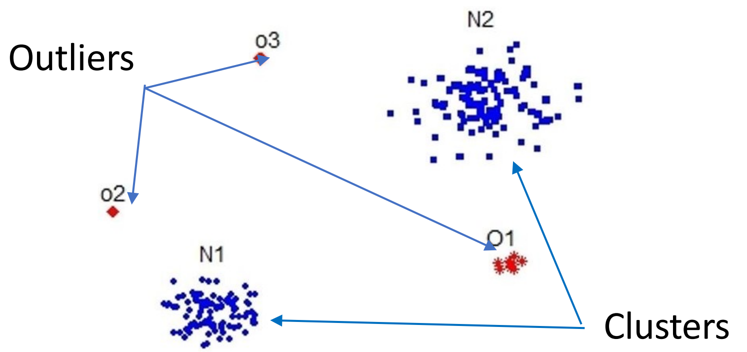


Representation Functions

Picks Centroids

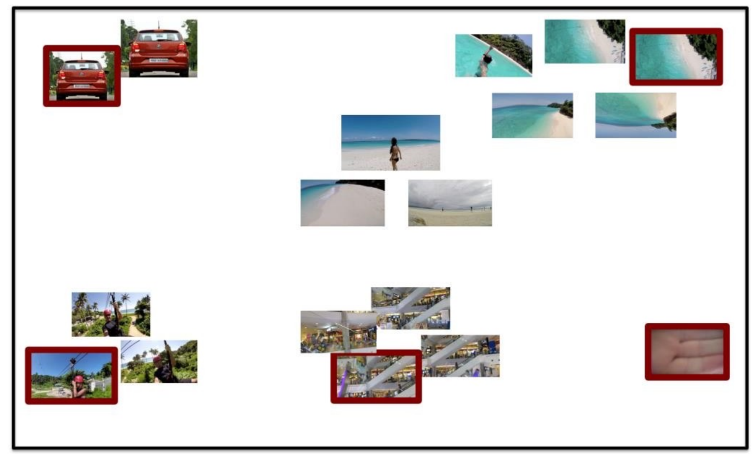
Iyer 2015, Kaushal et al 2019, Tschiatchek et al2014, …

Diversity Functions: Dispersion



Dispersion Min	$\min_{k,l \in X, k \neq l} d_{kl}$
Dispersion Sum	$\sum_{k,l \in X} d_{kl}$
Dispersion Min-Sum	$\sum_{k \in X} \min_{l \in X} d_{kl}$

Dispersion Sum and Dispersion Min Not Submodular!

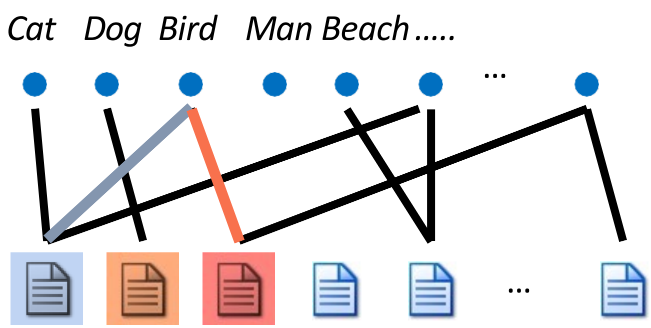


Diversity Functions

Picks items as different as possible!

Dasgupta et al 2013, Chakraborty et al 2015

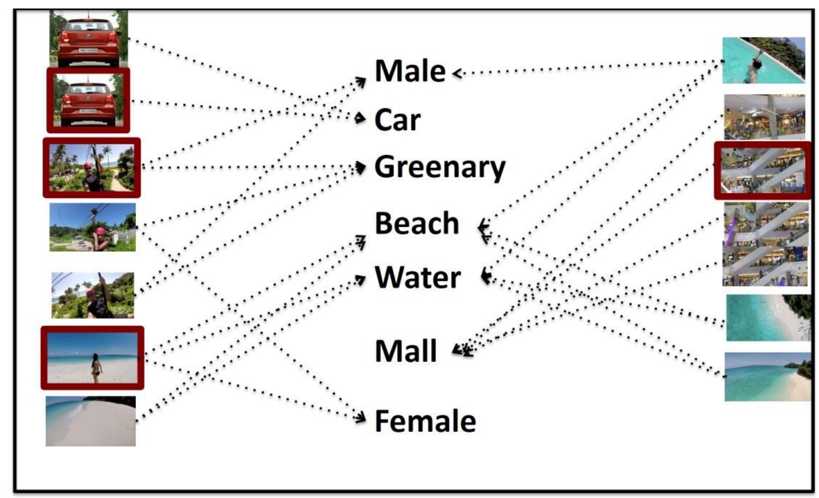
Coverage Functions



Set Cover Function

$$f(X) = w(\cup_{i \in X} U_i),$$

Concepts Covered by Instance i

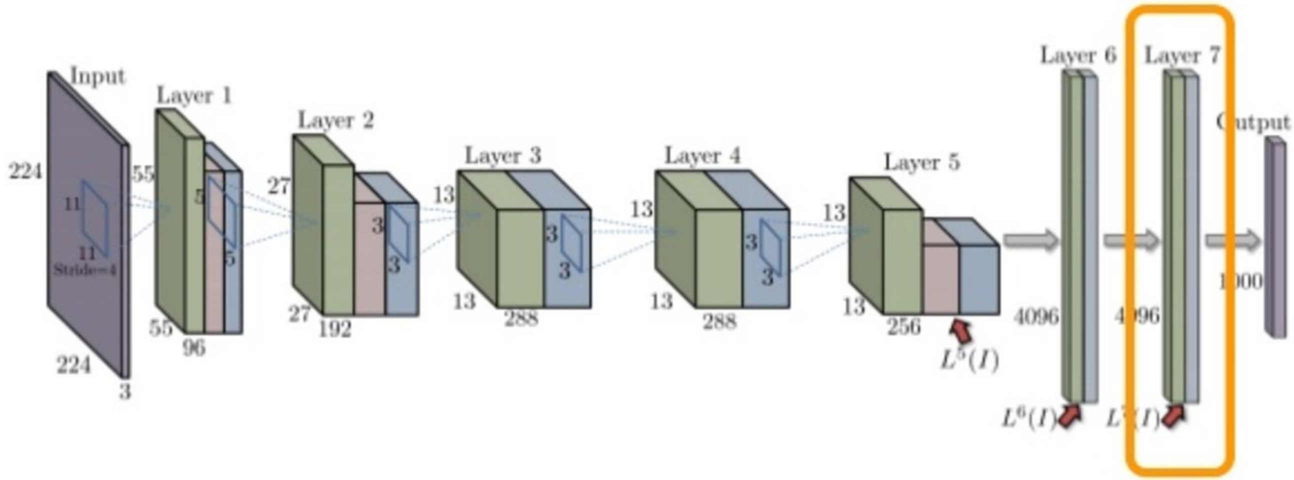


Coverage Functions

Select instances which “cover” all concepts

Wolsey et al 1982, ...

Feature Based Functions



Achieve Uniformity in Feature Coverage

Feature Based Functions

$$f_{\text{fea}}(S) = \sum_{u \in \mathcal{U}} g(m_u(S)).$$

Wei-Iyer et al 2014 ...

Total Contribution of Feature u in the Set of Images S

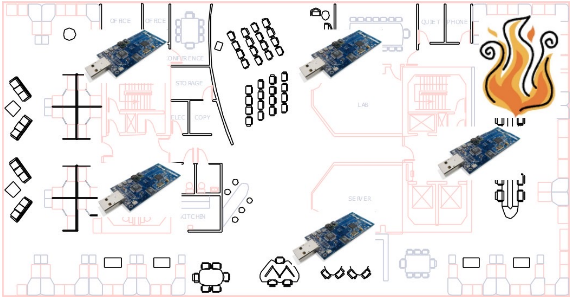
g is concave function

Information Functions

$X_1, \ldots, X_n$ discrete random variables: $X_e \in \{1, \ldots, m\}$

$F(S) = H(X_S) =$ joint entropy of variables indexed by S

$$H(X_e) = \sum_{x \in \{1,\ldots,m\}} P(X_e = x) \log P(X_e = x)$$



$$\boxed{A \subset B, e \notin B \qquad F(A \cup e) - F(A) \geq F(B \cup e) - F(B)??}$$

$$\begin{aligned} H(X_{A \cup e}) - H(X_A) &= H(X_e | X_A) \\ &\leq H(X_e | X_B) \qquad \text{“information never hurts”} \\ &= H(X_{B \cup e}) - H(X_B) \end{aligned}$$

discrete entropy is submodular!

Entropy
Mutual Information
Information Gain
...

Krause et al 2008, ...

Master Optimization Problem

Set Function   Selected set

$$\max_{\mathcal{A} \subseteq \mathcal{V}} F(\mathcal{A})$$

Selection cost   Budget

$$\text{subject to } C(\mathcal{A}) \leq B$$

F = Monotone Submodular,
Non Monotone Submodular,
Dispersion Functions,
....

F Models:

- Diversity
- Representation
- Coverage
- Information
- Importance
- …

We shall study this and variants of this Master Optimization Problem!

Monotone Submodular Maximization

$$\max_{S} \; F(S) \;\; \text{s.t.} \;\; |S| \leq k$$

What is the Constraint?
C(S) = |S|

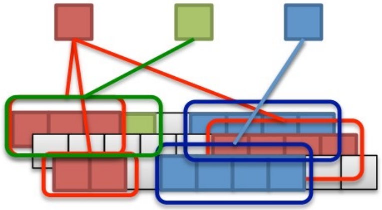
- greedy algorithm:

$S_0 = \emptyset$

for *i* = *0, …, k-1*

$$e^* = \arg \max_{e \in \mathcal{V} \setminus S_i} F(S_i \cup \{e\})$$

$$S_{i+1} = S_i \cup \{e^*\}$$

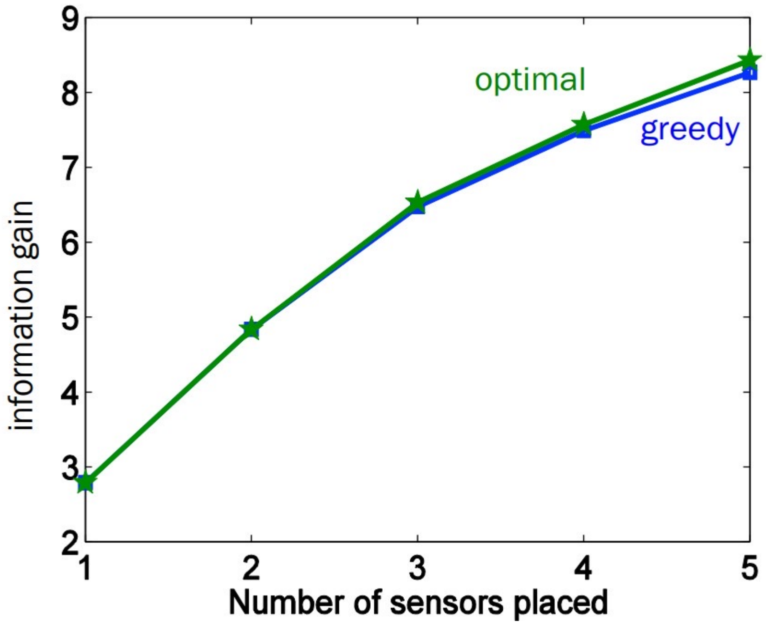


How "good" is S_k ?

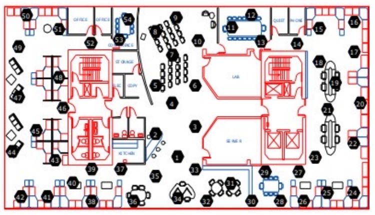
Approximation Guarantee!

How good is Greedy in Practice?

empirically:



sensor placement



How good is Greedy in Theory?

$$\max_S \; F(S) \; \text{ s.t. } |S| \le k$$

Theorem *(Nemhauser, Fisher, Wolsey `78)*

F monotone submodular, S_k solution of greedy. Then

$$F(S_k) \;\geq\; \left(1 - \frac{1}{e}\right) F(S^*)$$

optimal solution

No Poly-time algorithm can do better than this in the worst case!

Proof (Nemhauser et al 1978)

Let:

- $A_i = (v_1, v_2, \ldots, v_i)$ be the the chain formed by the greedy algorithm, as defined above
- $A^* = (v_1^*, v_2^*, \ldots, v_k^*)$ be the optimal solution, in an arbitrary order
- f be a monotone submodular function. Let $f \geq 0$ (*Update on 04/25/2019: I thought this was w.l.o.g., but Andrey Kolobov pointed out that we actually need f to be non negative*)
- $OPT = f(A^*)$, the value of the optimal solution.

We will prove that

$$f(A_k) \geq (1 - 1/e)OPT$$

Source: https://homes.cs.washington.edu/~marcotcr/blog/greedy-submodular/

Proof (Nemhauser et al 1978)

For all $i \leq k$, we have:

$$\begin{aligned}
f(A^*) &\leq f(A^* \cup A_i) && \text{Monotonicity} \\
&= f(A_i) + \sum_{j=1}^{k} \Delta(v_j^* | A_i \cup \{v_1^*, v_2^*, \ldots, v_{j-1}^*\}) && \\
&\leq f(A_i) + \sum_{z \in A^*} \Delta(z | A_i) && \text{Using submodularity} \\
&\leq f(A_i) + \sum_{z \in A^*} \Delta(v_{i+1} | A_i) && v_{i+1} = argmax_{v \in V \setminus A_i} \Delta(v | A_i) \\
&= f(A_i) + k\Delta(v_{i+1} | A_i) &&
\end{aligned}$$

Rearranging the terms, we have proved that

$$\Delta(v_{i+1} | A_i) \geq \frac{1}{k}(OPT - f(A_i))$$

Source: https://homes.cs.washington.edu/~marcotcr/blog/greedy-submodular/

Proof (Nemhauser et al 1978)

Part I

Now we define $\delta_i = OPT - f(A_i)$. This implies
$\delta_i - \delta_{i+1} = f(A_{i+1}) - f(A_i) = \Delta(v_{i+1}|A_i)$

Plugging this into our previous equation, we have:

$$\Rightarrow \quad \delta_i - \delta_{i+1} \geq \frac{1}{k}(\delta_i)$$

$$\Rightarrow \quad \delta_{i+1} \leq (1 - \frac{1}{k})\delta_i$$

Part II

$$\Rightarrow \quad \delta_k \leq \left(1 - \frac{1}{k}\right)^k \delta_0$$

$$\Rightarrow \quad \delta_k \leq \left(1 - \frac{1}{k}\right)^k OPT \leq \frac{1}{e} OPT$$

$$\Rightarrow \quad OPT - f(A_k) \leq \frac{1}{e} OPT$$

$$\Rightarrow \quad f(A_k) \geq \left(1 - \frac{1}{e}\right) OPT$$

Source: https://homes.cs.washington.edu/~marcotcr/blog/greedy-submodular/

Monotone Submodular – Budget Constraints

$$\max \quad F(S) \;\; \text{s.t.} \; \sum_{e \in S} c(e) \leq B$$

1. run greedy: S_{gr}
2. run a modified greedy: S_{mod}

$$e^* = \arg\max \frac{F(S_i \cup \{e\}) - F(S_i)}{c(e)}$$

3. pick better of S_{gr}, S_{mod}

➔ approximation factor:

$$\frac{1}{2}\left(1 - \frac{1}{e}\right)$$

even better but less fast: partial enumeration *(Sviridenko, 2004)* or filtering *(Badanidiyuru & Vondrák 2014)*

Sviridenko 2004:

- Run the cost-sensitive greedy algorithm starting with all possible initial sets {i,j,k}
- $O(n^3)$ initial complexity
- (1 – 1/e) approximation!

Sviridenko 2004, Leskovec et al 2007

Summary: Greedy Algorithm Framework

Monotone Submodular Function

$$\max_{S \subseteq V, c(S) \leq \mathcal{B}} f(S)$$

Cost of Summary Subset S (e.g. size)

Problem Formulation

Initialization $S \leftarrow \emptyset$.
repeat
Pick an element $v^* \in \operatorname{argmax}_{v \in V \setminus S} \frac{f(v \cup S) - f(S)}{c(v)}$
Update $S \leftarrow S \cup v^*$
until Reaching the budget, i.e., $c(S) > \mathcal{B}$

Greedy Algorithm

Non-Monotone Submodular Functions

$$\max_S \; F(S) \;\text{ s.t. } |S| \leq k$$

Start with $Y_0 = \emptyset$

for $i = 1$ *to* k **do**

  Let $M_i = \text{argmax}_{X \subseteq V \setminus Y_{i-1}, |X|=k} \sum_{v \in X} f(v|Y_{i-1})$;

  Choose y as a uniformly random element in M_i;

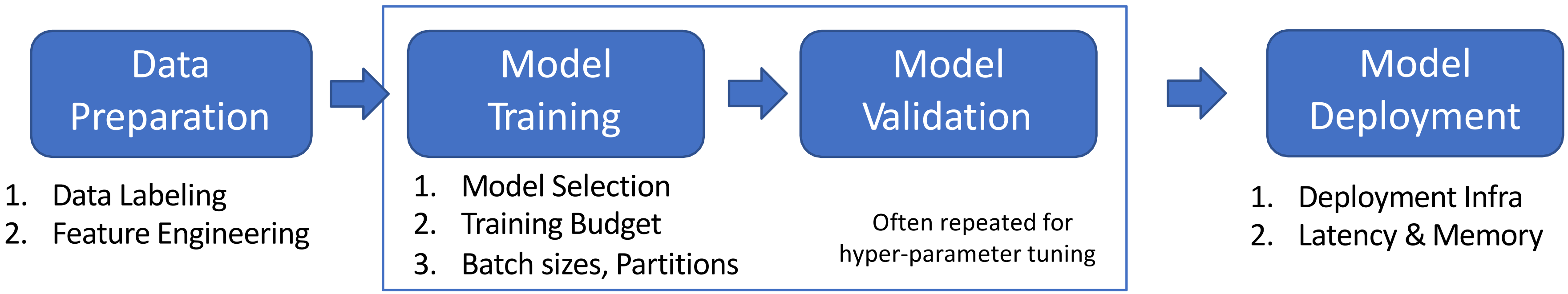
  $Y_i = Y_{i-1} \cup y$;

return Y_k.

Theorem (Buchbinder et al 2014): The Randomized Greedy Algorithm achieves a 1/e approximation guarantee for Non-Monotone Submodular Maximization subject to cardinality constraints!

Data subset selection

Make ML Data Efficient and Robust



Production Systems Constraints

1. Data Labeling => Time Consuming, Expensive, Noisy
2. Feature Selection => Latency & Memory
3. Model Training => Compute Intensive and Time Consuming
4. Hyper-Parameter Tuning/NAS => Very Time Consuming
5. Distribution Shift => Deployment vs Training

Can we train Models under these constraints without sacrificing on accuracy?

Data Subset Selection Setup

A Machine Learning model characterized by model parameters θ

Training Data: $\{(x_i, y_i), i \in \mathcal{U}\}$ Training log-likelihood function: $LL_T(\theta, \mathcal{U})$

Training a machine learning model often reduces to finding the parameters that maximizes a log-likelihood function for given training data empirically.

$$\theta^* = \underset{\theta}{\operatorname{argmax}}\, LL_T(\theta, \mathcal{U})$$

Validation Data: $\{(x_i, y_i), i \in \mathcal{V}\}$ Validation log-likelihood function: $LL_V(\theta, \mathcal{V})$

Goal: Select a subset $S \subseteq \mathcal{U}$ such that the resulting model performs the **best**!

Requirements for optimal subset selection

1. The subset selection algorithm needs to be as fast as possible.
   - Subset Selection time <<<< Full training time

Example: Subset selection algorithm with negligible time complexity

Training on 10 % Subset → 10x Faster training

2. Theoretical guarantees of subset selection algorithm.
   - Can we show theoretical guarantees for subset selection algorithms?

Approaches for Data Subset Selection

- Several different kinds of approaches studied in literature:
  - Approach 1: Use Submodular Functions as proxy functions for data subset selection
  - **Approach 2: Choose data subset which approximates the gradient of the entire dataset**
  - Approach 3: Choose data subset which approximates the performance on full training dataset (or validation set) as a bi-level optimization!
  - Approach 4: Choose data subset which minimizes a suitable divergence (e.g. KL divergence) between the distribution induced by the subset and full data!
- Types of Data Selection
  - Supervised (Using the labels)
  - Unsupervised (No access to labels)
  - Validation based (Access to a validation set for focusing on generalization)

Idea: Gradient Matching/ CoreSets

Can we obtain a weighted gradient of a **subset** of points that approximates the full gradient?

$$\sum_{i \in X_t} w_i^t \nabla_\theta L_T^i(\theta) \approx \nabla_\theta L(\theta)$$

Sivasubramaniam & Killamsetty et. al. 2021, Mirzasoleiman et al 2020

Gradient Matching: Main Idea

The theorem indicates that an effective data selection algorithm should try to have a low error $\mathrm{Err}(\mathbf{w}^t, X_t, L, L_T, \theta_t)$ for $t = 1, \cdots, T$. Thus, we can pose the problem as,

$$\begin{aligned}\mathbf{w}^t, X_t &= \min_{\mathbf{w}, X:|X|\leq k} \mathrm{Err}(\mathbf{w}, X, L, L_T, \theta_t) \\ &= \min_{\mathbf{w}, X:|X|\leq k} \|\sum_{i\in X_t} w_i^t \nabla_\theta L_T^i(\theta_t) - \nabla_\theta L(\theta_t)\|\end{aligned}$$

Sivasubramaniam & Killamsetty et. al. 2021

Directly Optimizing Gradient Error: GradMatch

Define the regularized version of our objective:

$$E_\lambda(X) = \min_{\mathbf{w}} \underbrace{\Big\| \sum_{i \in X_t} w_t^i \nabla_\theta L_T^i(\theta_t) - \nabla_\theta L(\theta_t) \Big\|^2 + \lambda ||\mathbf{w}^t||^2}_{E_\lambda(X_t, \mathbf{w}^t)}$$

This problem can be solved efficiently using Orthogonal Matching Pursuit (OMP) described as,

1. Find projection of $r = \nabla_\theta L_T^i(\theta_t)$ for each $i \in W$ along $\nabla_\theta L(\theta_t)$ and chose the i with whom projection is maximum and add it X
2. Solve linear regression problem to find w_t^i for $i \in X$s.
3. Set $r = \nabla_\theta L(\theta_t) - \sum_{i \in X_t} w_t^i \nabla_\theta L_T^i(\theta_t)$
4. Repeat the steps with new r until the $|r| < \epsilon$ or $|X| < k$(budget)
5. Return X, w_t

Sivasubramaniam & Killamsetty et. al. 2021

Orthogonal Matching Pursuit

The OMP algorithm

Algorithm 1: OMP$(\mathbf{A}, \mathbf{b})$

Input: $\mathbf{A}, \mathbf{b}$

Result: $\mathbf{x}_k$

Initialization $\mathbf{r}_0 = \mathbf{b}$, $\Lambda_0 = \varnothing$;

Normalize all columns of $\mathbf{A}$ to unit L_2 norm;

Remove duplicated columns in $\mathbf{A}$;

for $k = 1, 2, \dots$ **do**

- Step-1. $\lambda_k = \underset{j \notin \Lambda_{k-1}}{\operatorname{argmax}} \left| \langle \mathbf{a}_j, \mathbf{r}_{k-1} \rangle \right|$;
- Step-2. $\Lambda_k = \Lambda_{k-1} \cup \{\lambda_k\}$;
- Step-3. $\mathbf{x}_k(i \in \Lambda_k) = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{A}_{\Lambda_k} \mathbf{x} - \mathbf{b}\|_2, \quad \mathbf{x}_k(i \notin \Lambda_k) = 0$;
- Step-4. $\hat{\mathbf{b}}_k = \mathbf{A}\mathbf{x}_k$;
- Step-5. $\mathbf{r}_k \leftarrow \mathbf{b} - \hat{\mathbf{b}}_k$;

end

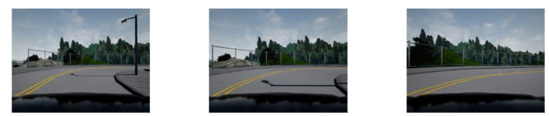
Convex DSS

Aim

- We study the problem of data efficient training of autonomous driving systems.

- Training using many frames on straight road sections may not be necessary. Frames at the turns turn out to be useful.

REDUNDANT

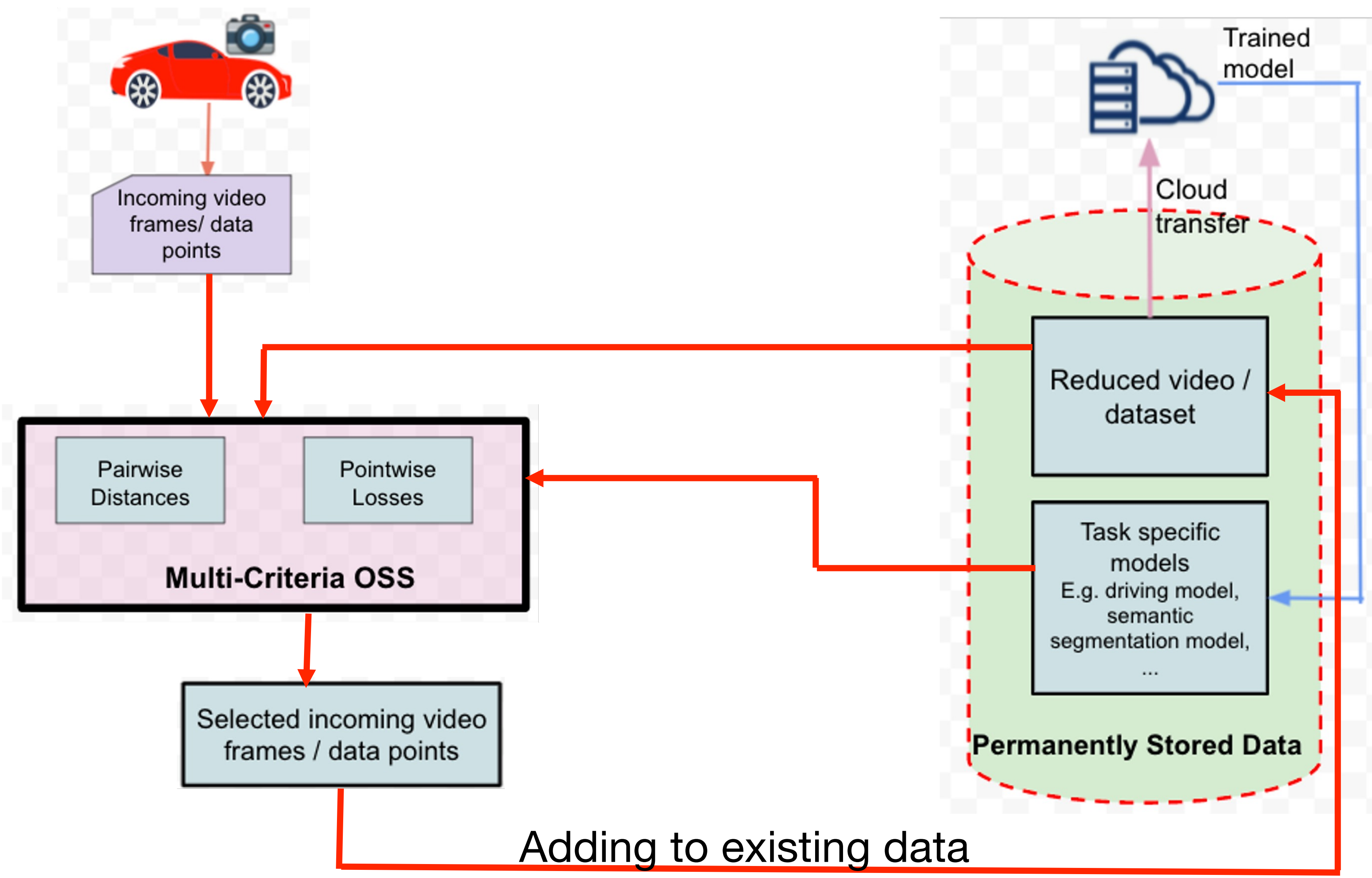


INFORMATIVE

Method	Train One-Turn	Test One-Turn
Uniform Skip	3/10	5/10

In the context of edge device deployment, multi-criteria online subset selection (OSS) framework can be useful in selecting informative frames, essential for an end-task.

Subset selection on Edge devices



High Level Idea

- Given a compression ratio, find out representatives which have the least dissimilarity with the left-out elements besides having the highest task-specific loss.

Problem Setup

- X_t: the set of incoming datapoints at time t (Size m)
- D: set of all data points (Size N)
- R_t: Reduced set of data at time t
- d_{ij}: Distance between data points I and j.
- z_{ij}: Indicator variable indicating that datapoint i is a representative for datapoint j.

Convex Subset Selection

- Original formulation in set notation:

$$\min_{\mathcal{S} \subseteq \mathcal{D}} \lambda |\mathcal{S}| + \sum_{j \in \mathcal{D}} \min_{i \in \mathcal{S}} d_{ij},$$

- Formulation using indicator random variables z_{ij}:

$$\min_{\{z_{ij}\}} \lambda \sum_{i \in \mathcal{D}} \mathrm{I}(\big\| \begin{bmatrix} z_{i1} & z_{i2} & \cdots & \end{bmatrix} \big\|_p) + \sum_{j \in \mathcal{D}} \sum_{i \in \mathcal{D}} d_{ij} z_i$$

Size regularizer

$$\text{s.t.} \quad z_{ij} \in \{0, 1\}, \ \sum_{i=1}^{N} z_{ij} = 1, \ \forall\, i, j \in \mathcal{D}.$$

- Convex relaxation:

$$0 \leq z_{ij} \leq 1$$

Online Subset Selection

- At time t:
  R_{t-1}: old set (denoted by superscript o)
  X_t: in the new set (denoted by superscript n)
  R_t: the new reduced set that we are trying to compute using z_{ij}
  $$R_t = R_{t-1} \cup \{i \in X_t | Z_{ij} = 1\}$$

- Revised formulation:

$$J'_{\text{enc}} \triangleq \sum_{i \in \mathcal{E}_o} \sum_{j \in \mathcal{D}_n} d_{ij}^{o,n} z_{ij}^{o,n} + \sum_{i \in \mathcal{D}_n} \sum_{j \in \mathcal{D}_n} d_{ij}^{n,n} z_{ij}^{n,n},$$

$$\begin{aligned} \min_{\mathcal{Z}'} \; & J'_{\text{enc}} + \lambda \sum_{i \in \mathcal{D}_n} \mathrm{I}(\big\| \begin{bmatrix} z_{i1}^{n,n} & z_{i2}^{n,n} & \cdots \end{bmatrix} \big\|_p) \\ \text{s.t.} \; & z_{ij}^{o,n},\; z_{ij}^{n,n} \in \{0, 1\}, \;\; \forall i,\; j, \\ & \sum_{i \in \mathcal{E}_o} z_{ij}^{o,n} + \sum_{i \in \mathcal{D}_n} z_{ij}^{n,n} = 1, \;\; \forall\, j \in \mathcal{D}_n, \end{aligned}$$

$$\boxed{\begin{gathered} \epsilon_o = R_{t-1} \\ D_n = X_t \end{gathered}}$$

High Level Idea

- Given a compression ratio, find out representatives which have the least dissimilarity with the left-out elements besides having the highest task-specific loss.

- Highest task-specific loss ensures having situational tasks needed to be learnt more by the model.

TMCOSS

Adopts a facility location objective involving multiple criteria

$$\min_{z^o_{ij}, z^n_{ij}} \mathcal{G}(z^o_{ij}, z^n_{ij}) s\,.\,t\,. \sum_{j=1}^{|R_t|} z^o_{i,j} + \sum_{j=1}^{m} z^n_{i,j} = 1; z^n_{i,j}, z^o_{i,j} \in [0,1]; \sum_{j=1}^{m} \|[z^n_{1,j} \ldots z^n_{m,j}]\|_p \le frac * m$$

Objective function

Constraint 1

Constraint 2

Compression Ratio

$z^o_{ij} = 1$ Denotes j from existing set o is a representative of element i from incoming set n

$z^n_{ij} = 1$ Denotes j from incoming set n is a representative of element i from incoming set n

$$\mathcal{G}(z^o_{ij}, z^n_{ij}) = \rho(\underbrace{\sum_{i=1}^{m}\sum_{j=1}^{|R_t|} z^o_{ij} d^o_{ij}(t) + \sum_{i,j=1}^{m} z^n_{ij} d^n_{ij}(t)}_{\text{Dissimilarity}}) - (1-\rho)(\underbrace{\sum_{j=1}^{|R_t|} S^o_j * L^o_j + \sum_{j=1}^{m} S^n_j * L^n_j}_{\text{Task specific Loss}})\ where, \underbrace{S^o_j = \frac{1}{\epsilon}\min(\epsilon, \sum_{i=1}^{m} z^o_{ij}), S^n_j = \frac{1}{\epsilon}\min(\epsilon, \sum_{i=1}^{m} z^n_{ij})}_{\text{Representative power of element j thresholded by } \epsilon}$$

Justification for thresholding

Theorem 1 *Let z^o_{ij} and z^n_{ij} be the optimal solution for formulation 1. A new frame $j \in X_{t+1}$ is selected as a representative frame for at least one incoming frame $i \in X_{t+1}$, i.e. $z^n_{ij} = 1$, only if BOTH these conditions hold:*

- *For some incoming frame $i \in X_{t+1}$, $Q^n_{ij} < Q^n_{ij'}$, for all $j' \in X_{t+1}$ and $j' \neq j$*
- *For some incoming frame $i \in X_{t+1}$, $Q^n_{ij} < \frac{\sum_{i'=1}^m z^o_{i',k} Q^o_{i'k} + \lambda \|[z^n_{1,j} \dots z^n_{m,j}]\|_p}{\|\mathbf{z}^n_j\|_1}$*

where $k = argmin_j \sum_{i=1}^m z^o_{i,j} Q^o_{i,j}$, and $\|\mathbf{z}^n_j\|_1 = \sum_{i'=1}^m z^n_{i'j}$

$\rho = 0$

Corollary 1.1 *Let z^o_{ij} and z^n_{ij} be the optimal solution for formulation 1. A new frame $j \in X_{t+1}$ is selected as a representative frame for at least one incoming frame $i \in X_{t+1}$, i.e. $z^n_{ij} = 1$, only if BOTH these conditions hold:*

- *$L^n_j > L^n_{j'}$ for all $j' \in X_{t+1}$ and $j' \neq j$*
- *$L^n_j > \frac{\sum_{i=1}^m z^o_{i,k} L^o_k - \lambda \|[z^n_{1,j} \dots z^n_{m,j}]\|_p}{\|\mathbf{z}^n_j\|_1}$*

where $k = argmin_j \sum_{i=1}^m z^o_{i,j} Q^o_{i,j}$, and $\|\mathbf{z}^n_j\|_1 = \sum_{i'=1}^m z^n_{i'j}$

Multi-criteria OSS (MCOSS)[1]

$$Q^n_{ij} = \rho d^n_{ij} - (1-\rho) L^n_j ; Q^o_{ij} = \rho d^o_{ij} - (1-\rho) L^o_j$$

$$\min_{z^o_{ij}, z^n_{ij}} \sum_{i=1}^{m} \sum_{j=1}^{|R_t|} z^o_{ij} Q^o_{ij} + \sum_{i,j=1}^{m} z^n_{ij} Q^n_{ij} + \lambda \sum_{j=1}^{m} \|[z^n_{1,j} \dots z^n_{m,j}]\|_p$$

$$s.t. \sum_{j=1}^{|R_t|} z^o_{i,j} + \sum_{j=1}^{m} z^n_{i,j} = 1, \ \forall i \in X_{t+1} z^n_{i,j}, z^o_{i,j} \in [0,1], \ \forall i,j$$

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