CS60021: Scalable Data Mining

Similarity Search and Hashing

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GENERALIZATION OF LSH

Locality sensitive hashing

- Originally defined in terms of a similarity function [C'02]

- Given universe U and a similarity $s: U \times U \rightarrow [0,1]$, does there exist a prob distribution over some hash family H such that

$$\Pr_{\mathrm{h} \in H}[h(x) = h(y)] = s(x,y)$$

$$s(x,y) = 1 \rightarrow x = y$$
$$s(x,y) = s(y,x)$$

Locality Sensitive Hashing

- Hash family H is *locality sensitive* if [Indyk Motwani]

$\Pr[h(x) = h(y)]$ is high if x is close to y

$\Pr[h(x) = h(y)]$ is low if x is far from y

- Not clear such functions exist for all distance functions

Hamming distance

- Points are bit strings of length d
- $H(x, y) = |\{i, x_i \neq y_i\}|$ $\quad S_H(x, y) = 1 - \frac{H(x,y)}{d}$
- Define a hash function h by sampling a set of positions
  - $x = 1011010001, y = 0111010101$
  - $S = \{1,5,7\}$
  - $h(x) = 100, h(y) = 100$

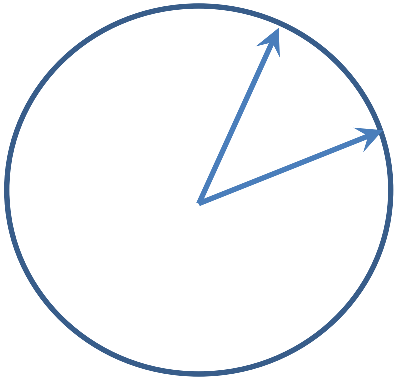
LSH for Hamming Distance

- The above hash family is locality sensitive, $k = |S|$

$$\Pr[h(x) = h(y)] = \left(1 - \frac{H(x,y)}{d}\right)^k$$

LSH for angle distance

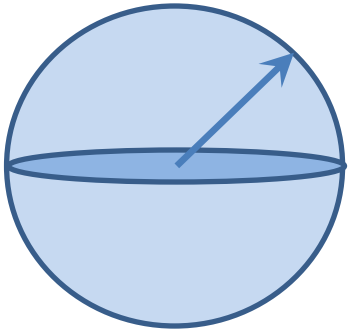
- x, y are unit norm vectors
- $d(x, y) = \cos^{-1}(x \cdot y) = \theta$
- $S(x, y) = 1 - \theta/\pi$



- Choose direction v uniformly at random
  - $h_v(x) = sign(v \cdot x)$
  - $\Pr[h_v(x) = h_v(y)] = 1 - \theta/\pi$

Aside: picking a direction u.a.r.

- How to sample a vector $x \in R^d, |x|_2 = 1$ and the direction is uniform among all possible directions



- Generate $x = (x_1, \dots . x_d), x_i \sim N(0,1)$ iid
- Normalize $\frac{x}{|x|_2}$
  - By writing the pdf of the d-dimensional Gaussian in polar form, easy to see that this is uniform direction on unit sphere

Which similarities admit LSH?

- There are various similarities and distance that are used in scientific literature
  - Encyclopedia of distances DL'11

- Will there be an LSH for each one of them?
  - Similarity is LSHable if there exists an LSH for it

[slide courtesy R. Kumar]

LSHable similarities

<u>Thm</u>: S is LSHable → 1 – S is a metric

$$d(x,y) = 0 \ \rightarrow x = y$$
$$d(x,y) = d(y,x)$$
$$d(x,y) + d(y,z) \geq d(x,z)$$

Fix hash function $h \in H$ and define

$$\Delta_h(A,B) = [h(A) \neq h(B)]$$
$$1 \ - \mathrm{S}(\mathrm{A},\mathrm{B}) = \Pr_h[\Delta_h(A,B)]$$

Also

$$\Delta_h(A,B) + \Delta_h(B,C) \geq \Delta_h(A,C)$$

Example of non-LSHable similarities

- $d(A,B) = 1 - s(A,B)$
- Sorenson-Dice : $s(A,B) = \frac{2|A \cap B|}{|A|+|B|}$
  - Ex: $A = \{a\}, B = \{b\}, C = \{a,b\}$
  - $s(A,B) = 0, s(B,C) = s(A,C) = \frac{2}{3}$

- Overlap: $s(A,B) = \frac{|A \cap B|}{\min(|A|,|B|)}$
  - $s(A,B) = 0, s(A,C) = 1 = s(B,C)$

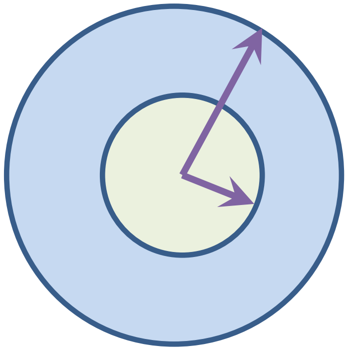
Gap Definition of LSH

- A family is (r, R, p, q) LSH if

IMRS'97, IM'98, GIM'99

$$\Pr_{h \in H}[h(x) = h(y)] \geq p \; if \; d(x, y) \leq r$$

$$\Pr_{h \in H}[h(x) = h(y)] \leq q \; if \; d(x, y) \geq R$$



Here $p > q$.

Gap LSH

- All the previous constructions satisfy the gap definition
  - Ex: for $JS(S,T) = \frac{|S \cap T|}{|S \cup T|}$

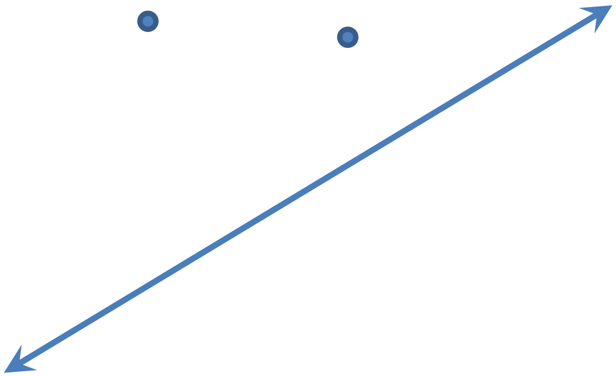
$$JD(S,T) \leq r \rightarrow JS(S,T) \geq 1 - r \rightarrow \Pr[h(S) = h(T)] = JS(S,T) \geq 1 - r$$

$$JD(S,T) \geq R \rightarrow JS(S,T) \leq 1 - R \rightarrow \Pr[h(S) = h(T)] = JS(S,T) \leq 1 - R$$

Hence is a $(r, R, 1-r, 1-R)$ LSH

L2 norm

- $d(x, y) = \sqrt{(}\sum_i (x_i - y_i)^2$
- $u =$ random unit norm vector, $w \in R$ parameter, $b \sim Unif\,[0, w]$



- $h(x) = \lfloor \frac{u \cdot x + b}{w} \rfloor$
- If $|x\ - y|_2 < \frac{w}{2},\ \Pr[h(x) = h(y)] \geq \frac{1}{3}$
- If $|x\ - y|_2 > 4w, \Pr[h(x) = h(y)] \leq \frac{1}{4}$

Solving the near neighbour

- (r, c) $-$near neighbour problem
  - Given query point q, return all points p such that $d(p, q) < r$ and none such that $d(p, q) > cr$
  - Solving this gives a subroutine to solve the “nearest neighbour”, by building a data-structure for each r , in powers of $(1 + \epsilon)$

How to actually use it?

- Need to amplify the probability of collisions for “near” points

Band construction

- AND-ing of LSH
  - Define a composite function $H(x) = (h_1(x), \ldots h_k(x))$
  - $\Pr[H(x) = H(y)] = \Pi_\mathrm{i} \Pr[h_i(x) = h_i(y)] = \Pr[h_1(x) = h_1(y)]^k$

- OR-ing
  - Create L independent hash-tables for $H_1, H_2, \ldots H_L$
  - Given query q, search in $\cup_j \, H_j(q)$

Example

	S_1	S_2	S_3	S_4
A	1	0	1	0
B	1	0	0	1
C	0	1	0	1
D	0	1	0	1
E	0	1	0	1
F	1	0	1	0
G	1	0	1	0

	S1	S2	S3	S3
h1	1	2	1	2
h2	2	1	3	1

	S1	S2	S3	S3
h3	3	1	2	1
h4	1	3	2	2

Why is this better?

- Consider q, y with $\Pr[h(q) = h(y)] = 1 - d(x, y)$
- Probability of not finding y as one of the candidates in $\cup_j H_j(q)$

$$1 - \left(1 - (1 - d)^k\right)^L$$

Creating an LSH

- Query x
- If we have a (r, cr, p, q) LSH

$\rho = \frac{\log(p)}{\log(q)}$ $L = n^{\rho}$ $k = \log(n) / \log\left(\frac{1}{q}\right)$

- For any y, with $|x - y| < r$,
  - Prob of y as candidate in $\cup_j H_j(x) \geq 1 - \left(1 - p^k\right)^L \geq 1 - \frac{1}{e}$
- For any z, $|x - z| > cr$,
  - Prob of z as candidate in any fixed $H_j(x) \leq q^k$
  - Expected number of such $z \leq Lq^k \leq L = n^{\rho}$
  - $\rho < 1$

Runtime

- Space used = $n^{1+\rho}$
- Query time = $n^{\rho} \times (k + d)$ [time for k-hashes & brute force comparison]

- We can show that for Hamming, angle etc, $\rho \approx \frac{1}{c}$
  - Can get 2-approx near neighbors with $O(\sqrt{n})$ neighbour comparisons

LSH: theory vs practice

- In order to design LSH in practice, the theoretical parameter values are only a guidance
  - Typically need to search over the parameter space to find a good operating point
  - Data statistics can provide some guidance.

Summary

- Locality sensitive hashing is a powerful tool for near neighbour problems
- Trades off space with query time
- Practical for medium to large datasets with fairly large number of dimensions
  - However, doesn't really work very well for sparse, very very high dimensional datasets
- LSH and extensions are an area of active research and practice

References:

- Primary references for this lecture
  - Modern Massive Datasets, Rajaraman, Leskovec, Ullman.
  - Survey by Andoni et al. (CACM 2008) available at www.mit.edu/~andoni/LSH