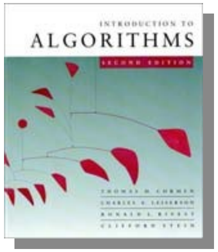
CS60020: Foundations of Algorithm Design and Machine Learning

Sourangshu Bhattacharya

DIVIDE AND CONQUER

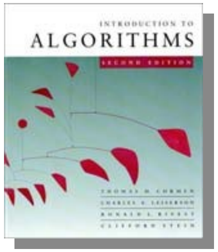


Fibonacci numbers

Recursive definition:

$$F_n = \begin{cases} 1 & \text{if } n = 0; \\ 2 & \text{if } n = 1; \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

0 1 1 2 3 5 8 13 21 34 L



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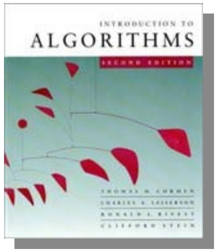
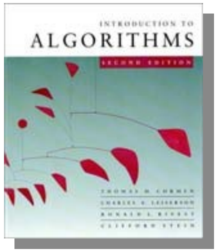
Naive recursive algorithm: $\Omega(\phi^n)$ (exponential time), where $\phi = (1+\sqrt{5})/2$ is the ***golden ratio***.



Computing Fibonacci numbers

Bottom-up:

- Compute $F_0, F_1, F_2, \ldots, F_n$ in order, forming each number by summing the two previous.
- Running time: $\Theta(n)$.



Computing Fibonacci numbers

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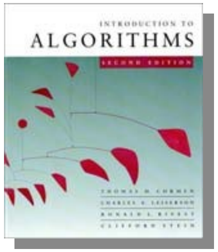
- Running time: $\Theta(n)$.

Naive recursive squaring:

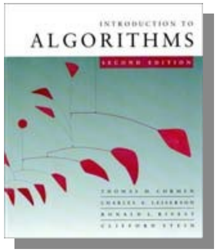
$F_n = \phi^n/\sqrt{5}$ rounded to the nearest integer.

- Recursive squaring: $\Theta(\lg n)$ time.
- This method is unreliable, since floating-point arithmetic is prone to round-off errors.



Recursive squaring

Theorem: $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n .$



Recursive squaring

Theorem: $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n .$

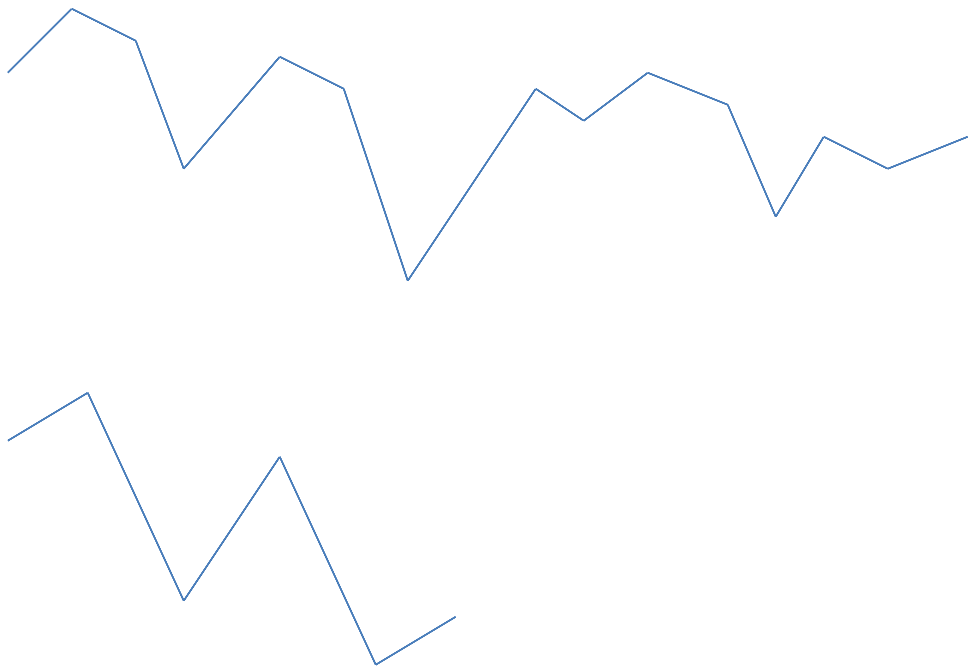
Algorithm: Recursive squaring.

Time = $\Theta(\lg n)$.

Maximum Subarray Problem

- You can buy a unit of stock, only *one* time, then sell it at a later date
  - Buy/sell at end of day
- Strategy: buy low, sell high
  - The lowest price may appear after the highest price
- Assume you know future prices
- Can you maximize profit by buying at lowest price and selling at highest price?

Buy lowest sell highest



Brute force

- How many buy/sell pairs are possible over n days?
- Evaluate each pair and keep track of maximum
- Can we do better?

Transformation

- Find sequence of days so that:
  - the net change from last to first is maximized
- Look at the daily change in price
  - Change on day *i*: price day *i* minus price day *i*-1
  - We now have an array of changes (numbers), e.g.
    12,-3,-24,20,-3,-16,-23,18,20,-7,12,-5,-22,14,-4,6
  - Find contiguous subarray with largest sum
    - **maximum subarray**
  - E.g.: buy after day 7, sell after day 11

Brute force again

- Trivial if only positive numbers (assume not)
- Need to check $O(n^2)$ pairs
- For each pair, find the sum
- Thus total time is …

Divide-and-Conquer

- A[low..high]
- Divide in the middle:
  – A[low,mid], A[mid+1,high]
- Any subarray A[i,..j] is
  (1) Entirely in A[low,mid]
  (2) Entirely in A[mid+1,high]
  (3) In both
- (1) and (2) can be found recursively

Divide-and-Conquer (cont.)

- (3) find maximum subarray that crosses midpoint
  - Need to find maximum subarrays of the form

    A[i..mid], A[mid+1..j], low <= i, j <= high
- Take subarray with largest sum of (1), (2), (3)

Divide-and-Conquer (cont.)

```
Find-Max-Cross-Subarray(A,low,mid,high)
      left-sum = -∞
      sum = 0
      for i = mid downto low
        sum = sum + A[i]
        if sum > left-sum then
                left-sum = sum
                max-left = i
      right-sum = -∞
      sum = 0
      for j = mid+1 to high
        sum = sum + A[j]
        if sum > right-sum then
                right-sum = sum
                max-right = j
return (max-left, max-right, left-sum + right-sum)
```

Time analysis

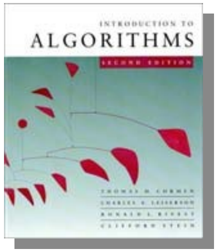
- Find-Max-Cross-Subarray: $O(n)$ time

- Two recursive calls on input size $n/2$

- Thus:

  $T(n) = 2T(n/2) + O(n)$

  $T(n) = O(n \log n)$

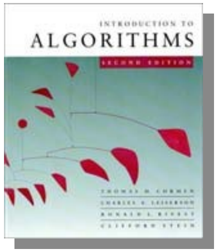


Matrix multiplication

$$\left.\begin{array}{ll}\textbf{Input:} & A = [a_{ij}],\ B = [b_{ij}]. \\ \textbf{Output:} & C = [c_{ij}] = A \cdot B.\end{array}\right\}\ i, j = 1, 2, \ldots, n.$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$



Standard algorithm

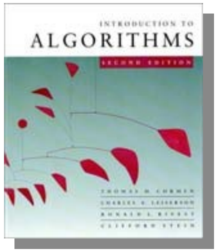
for $i \leftarrow 1$ **to** n

    do for $j \leftarrow 1$ **to** n

        do $c_{ij} \leftarrow 0$

            for $k \leftarrow 1$ **to** n

                do $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$



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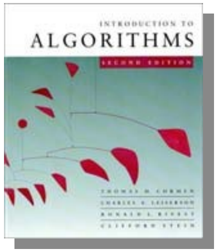
Running time $= \Theta(n^3)$



Divide-and-conquer algorithm

IDEA:

$n \times n$ matrix = 2×2 matrix of $(n/2) \times (n/2)$ submatrices:

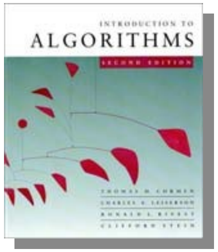
$$\left[\begin{array}{c|c} r & s \\ \hline t & u \end{array}\right] = \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right] \cdot \left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right]$$

$$C \quad = \quad A \quad \cdot \quad B$$

$$\left.\begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dg \\ u &= cf + dh \end{aligned}\right\}$$

8 mults of $(n/2) \times (n/2)$ submatrices

4 adds of $(n/2) \times (n/2)$ submatrices



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<u>*recursive*</u>

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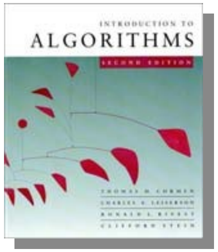


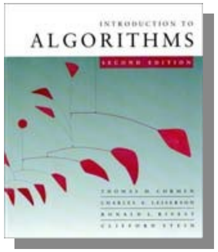
Analysis of D&C algorithm

$$T(n) = 8\, T(n/2) + \Theta(n^2)$$

 # submatrices

submatrix size

work adding submatrices



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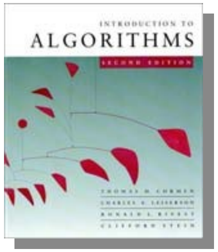
$$n^{\log_b a} = n^{\log_2 8} = n^3 \quad \Rightarrow \quad \textsc{Case 1} \quad \Rightarrow \quad T(n) = \Theta(n^3).$$



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 # submatrices

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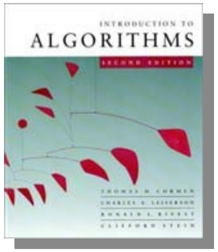
work adding submatrices

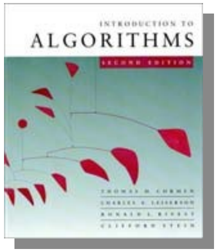
$$n^{\log_b a} = n^{\log_2 8} = n^3 \;\Rightarrow\; \text{CASE 1} \;\Rightarrow\; T(n) = \Theta(n^3).$$

No better than the ordinary algorithm.



Strassen's idea

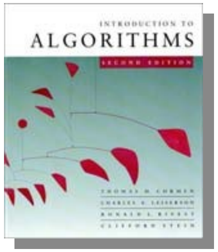
- Multiply 2×2 matrices with only 7 recursive mults.



Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$P_1 = a \cdot (f - h)$$
$$P_2 = (a + b) \cdot h$$
$$P_3 = (c + d) \cdot e$$
$$P_4 = d \cdot (g - e)$$
$$P_5 = (a + d) \cdot (e + h)$$
$$P_6 = (b - d) \cdot (g + h)$$
$$P_7 = (a - c) \cdot (e + f)$$



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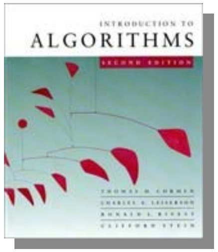
P_5 &= (a + d) \cdot (e + h) \\
P_6 &= (b - d) \cdot (g + h) \\
P_7 &= (a - c) \cdot (e + f)
\end{aligned}
$$

$$
\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
s &= P_1 + P_2 \\
t &= P_3 + P_4 \\
u &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$



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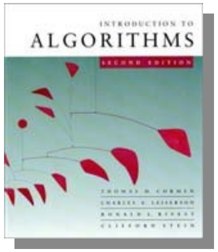
$$P_7 = (a - c) \cdot (e + f)$$

$$r = P_5 + P_4 - P_2 + P_6$$
$$s = P_1 + P_2$$
$$t = P_3 + P_4$$
$$u = P_5 + P_1 - P_3 - P_7$$

7 mults, 18 adds/subs.
Note: No reliance on commutativity of mult!



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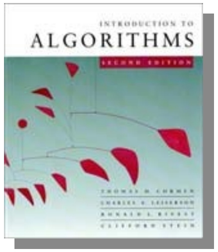
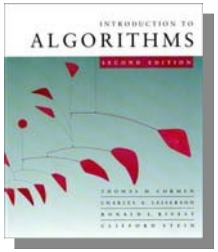
\end{aligned}
$$

$$
\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
&= (a + d)(e + h) \\
&\quad + d(g - e) - (a + b)h \\
&\quad + (b - d)(g + h) \\
&= ae + ah + de + dh \\
&\quad + dg - de - ah - bh \\
&\quad + bg + bh - dg - dh \\
&= ae + bg
\end{aligned}
$$



Strassen's algorithm

1. ***Divide:*** Partition A and B into $(n/2)\times(n/2)$ submatrices. Form terms to be multiplied using $+$ and $-$.
2. ***Conquer:*** Perform 7 multiplications of $(n/2)\times(n/2)$ submatrices recursively.
3. ***Combine:*** Form C using $+$ and $-$ on $(n/2)\times(n/2)$ submatrices.



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