Homomorphism in Formal Languages.

A homomorphism is a map $h: \Sigma^* \to \Gamma^*$
s.t. $\forall x, y \in \Sigma^*$, $h(xy) = h(x) \cdot h(y)$ — ①
& $h(\epsilon) = \epsilon$ — ②

Note, $|h(\epsilon)| = |h(\epsilon \cdot \epsilon)|$
$= |h(\epsilon) \cdot h(\epsilon)|$ by ①
$= |h(\epsilon)| + |h(\epsilon)|$
$\Rightarrow |h(\epsilon)| = 0$
$\Rightarrow h(\epsilon) = \epsilon$
$\therefore$ ① $\Rightarrow$ ②

Regular Languages are closed under homomorphism

i) if R is a regex, $h(R)$ is also regex.
↳ inductive proof possible.

ii) if L is regular, $h(L)$ is also regular.

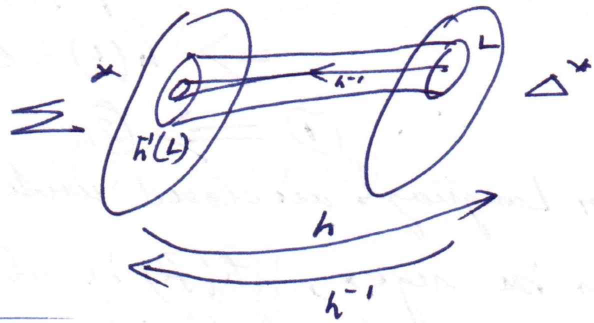
iii) $h(L_1 \cup L_2) = h(L_1) \cup h(L_2)$
$h(L_1 \cdot L_2) = h(L_1) \cdot h(L_2)$
$h(L^*) = (h(L))^*$

Inverse homomorphism

if $h: \Sigma^* \to \Delta^*$ is a homomorphism,
& $L \subseteq \Delta^*$ is some language,
$h^{-1}(L) = \{ w \in \Sigma^* \mid h(w) \in L \}$

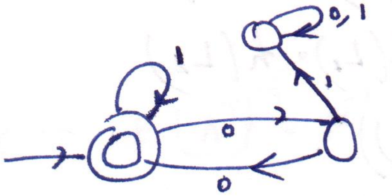
- If L is regular, $h^{-1}(L)$ is regular.

In general, $h(h^{-1}(L)) \subseteq L \subseteq h^{-1}(h(L))$



Ex $\Sigma = \{a, b\}$, $\Delta = \{0, 1\}$

$L = (00 + 1)^*$



Let $h: \Sigma^* \to \Delta^*$
$h(a) = 01$
$h(b) = 10$
$h^{-1}(1001) = \{ba\}$
$h^{-1}(010110) = \{aab\}$

note that,
inverse can generate sets,
h is a fn., not a bijection.

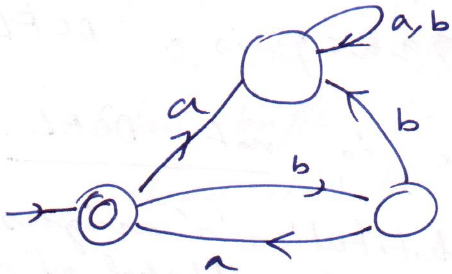
check, $h(h^{-1}(L)) \subsetneq L$

in this case . .

∴ in general ~~neither~~ containment

need not have ~~not~~ equality.

Automaton for $h^{-1}(L)$



$h^{-1}(L) = (ba)^*$

$h(h^{-1}(L)) = (1001)^*$

~~$h^{-1}\left((00+11)^*\right)$~~

~~00~~

formal construction :→

Let M accept $L \subseteq \Delta^*$.

$h : \Sigma^* \to \Delta^*$

$\uparrow M' \quad \uparrow M$

$M = \langle Q, \Delta, \delta, q_0, F \rangle$

Let $M' = \langle Q', \Sigma, \delta', q_0', F' \rangle$ where,

$Q' = Q, \; q_0' = q_0, \; F' = F$

& $\delta'(q_1, a) = q_2$

iff $\hat{\delta}(q_1, h(a)) = q_2$

observe, M' accepts $h^{-1}(L)$ since,

$\forall w \in \Sigma^*, \; \hat{\delta}'(q_0, w) = \hat{\delta}(q_0, w)$

CFL closure properties

① closure of CFLs under reversal

for ever production $X \longrightarrow v$

make the rule $X \longrightarrow v^R$

② nonclosure of DCFLs under ~~reveral~~ reversal.

$L = \{ba^m b^n c^k \mid m \neq n\} \cup \{ca^m b^n c^k \mid n \neq k\}$

over $\Sigma = \{a, b, c\}$ is a DCFL.

The reverse is not DCFL.

③ If a substitution 's' assigns a CFL to every symbol in the alphabet of a CFL L, then $s(L)$ is a CFL.

Consequence → closure under union, concatenation, star, homomorphism.

CFL closed under inverse homomorphism.

i/p string — — -- -.
↓ need to read as
$h() \, h() \, h() \, h()$

If PDA P accepts L, P' needs to accept $h^{-1}(L)$

$h : \Sigma^* = \{a, b\}^* \longrightarrow \Gamma^* = \{0, 1\}^*$

consider i/p σ' for P'.

$\sigma' \in L(P')$ if $\exists \, \sigma \in L(P)$ s.t. $\sigma' = h^{-1}(\sigma)$

$\Rightarrow \sigma = h(\sigma')$

$\therefore$ P' will scan i/p σ',

& keep applying h.

$P' \rightarrow$ i) read $\sigma'[i]$

ii) compute $h(\sigma'[i])$

put this in a buffer.

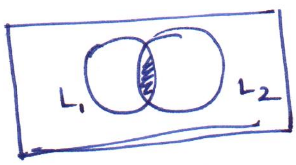
iii) apply rules of P sequentially on the buffer.

when buffer is empty, load it with $h(\sigma'[i+1])$

Another property

CFLs are not closed under set difference.

$$L_1 \cap L_2 = L_1 \setminus (L_1 \setminus L_2)$$



had CFLs been closed under diff, they would have been closed under intersection.