Indian Institute of Technology Kharagpur
Department of Computer Science and Engineering

Foundations of Computing Science (CS60005)		**Autumn Semester, 2023-2024**
Mid-Semester Examination	**Date: 20-Sep-2023, 2:00 PM – 4:00 PM**	**Marks: 60**

Instructions:

- Write your answers in the answer booklet provided to you in the examination hall.
- Answer ALL five questions mentioning the question numbers clearly.
- Be brief and precise. Write the answers for all parts of a question together.
- State any results you use or assumptions you make.
- Keep all your answers/proofs mathematically precise and to the point.
- Sketchy proofs or claims without reasoning receive no credit.

1. (Based on "The Adventure of Silver Blaze" by Sir Arthur Conan Doyle).
A prize-winning racehorse named Silver Blaze has been stolen from a stable, and a bookmaker named Fitzroy Simpson has been arrested as the prime suspect by Inspector Gregory. Your task is to (logically) find who stole Silver Blaze on behalf of Sherlock Holmes. The premises are as follows.

F_1**:** *The horse was stolen either by Fitzroy or by its trainer John Straker.*

F_2**:** *The thief had to have entered the stable the night of the theft.*

F_3**:** *If a stranger enters the stable, the dog barks.*

F_4**:** *Fitzroy was a stranger.*

F_5**:** *The dog did not bark.*

Your goal is to (logically) find the reason for the murder. Please frame the above arguments logically (using propositional logic) and formally derive the solution. Present your answer as asked in the following parts.

(a) Write all propositions with English meaning (statements) that you have used.

Solution: We use 7 propositions:

tf**:** Fitzroy is the thief

ts**:** Straker is the thief

ef**:** Fitzroy entered the stable the night of the theft

es**:** Straker entered the stable the night of the theft

d**:** The dog barked

sf**:** Fitzroy is a stranger

ss**:** Straker is a stranger

(b) Build suitable propositional logic formula to encode each of the five statements above.

Solution:

F_1**:** $(tf \vee ts) \wedge \neg(tf \wedge ts)$

F_2**:** $(tf \to ef) \wedge (ts \to es)$

F_3**:** $(ef \wedge sf \to d) \wedge (es \wedge ss \to d)$

F_4**:** sf

F_5**:** $\neg d$

(c) Show all deduction steps (with the name of the rules you apply) to derive the goal.

Solution:

$$\frac{(ef \wedge sf \to d) \wedge (es \wedge ss \to d)}{\therefore (ef \wedge sf \to d)} \text{ Conjunction Elimination}$$

$$\frac{\begin{array}{l}(ef \wedge sf \to d)\\ sf\end{array}}{\therefore \neg(ef \wedge sf)} \text{ Modus Tollens}$$

$$\frac{\begin{array}{l}\neg(ef \wedge sf)\\ sf\end{array}}{\therefore \neg ef} \text{ Disjunctive Elimination}$$

$$\frac{\begin{array}{l}tf \to ef\\ \neg ef\end{array}}{\therefore \neg tf} \text{ Modus Tollens}$$

$$\frac{\begin{array}{l}tf \vee ts\\ \neg tf\end{array}}{\therefore ts} \text{ Disjunction Elimination}$$

(d) Conclude who stole Silver Blaze.

Solution: John Straker stole Silver Blaze.

$\boxed{2+2.5+6+1.5 = 12}$

2. Prove the following.

(a) $\mathbb{R} \sim \mathbb{R} \times \mathbb{R}$.

Solution: We know $\mathbb{R} \sim [0,1)$ and so $\mathbb{R} \times \mathbb{R} \sim [0,1) \times [0,1)$. Let $A = [0,1) \setminus \mathbb{Q}$. Then $A \sim [0,1)$ since $\mathbb{Q}$ is countable. Now consider the map $f : [0,1) \to [0,1) \times [0,1)$ defined as $f(x) = (x, x)$. It is easy to see that f is injective and consequently $[0,1) \times [0,1)$ is at least as large as $[0,1)$. Now define $g : A \times A \to A$ as follows: for $x = 0.x_1x_2\cdots, y = 0.y_1y_2\cdots \in A$ define $g(x,y) = 0.x_1y_1x_2y_2\cdots$. The function g is injective. So A is atleast as large as $A \times A$ thus implying that $[0,1) \times [0,1) \sim [0,1)$.

(b) Let A, B be two sets with $A \cap B = \emptyset$. Then $2^A \times 2^B \sim 2^{A\cup B}$. Do not make any assumptions about the cardinalities of A and B.

Solution: Define a function $f : 2^A \times 2^B \to 2^{A\cup B}$ as follows: for any subsets $S \subseteq A$ and $T \subseteq B$, define $f(S,T) = S \cup T$. Suppose that $f(S_1,T_1) = f(S_2,T_2)$. We have

$$\begin{aligned} f(S_1,T_1) &= f(S_2,T_2)\\ S_1 \cup T_1 &= S_2 \cup T_2\\ (S_1 \cup T_1) \cap A &= (S_2 \cup T_2) \cap A\\ S_1 \cap A &= S_2 \cap A\\ S_1 &= S_2 \end{aligned}$$

The fourth line follows from the fact that $T_1 \cap A = T_2 \cap A = \emptyset$ since both T_1, T_2 are subsets of B. Similarly, we can take intersection with B and show that $T_1 = T_2$. Therefore f is injective.

Let $U \subseteq A \cup B$. Since A, B are disjoint, $U \cap A \subseteq A$ and $U \cap B \subseteq B$. Let $S = U \cap A$ and $T = U \cap B$. Then $f(S,T) = S \cup T = U \cap (A \cup B) = U$. Therefore every element of $2^{A\cup B}$ has a preimage implying that f is a surjection. It follows that f is a bijection and hence $2^A \times 2^B \sim 2^{A\cup B}$.

$\boxed{6+6 = 12}$

3. Let $G = (V, E)$ be an undirected graph. For vertices $u, v \in V$, a *path* connecting u and v is a sequence of vertices $u_0 (= u), u_1, u_2, \ldots, u_k (= v)$ such that for every $i = 0, 1, \ldots, k-1$, the edge $\{u_i, u_{i+1}\} \in E$. Define a relation $\mathcal{R}$ on $V \times V$ as follows: $(u, v) \in \mathcal{R}$ if either $u = v$ or there is a path in G connecting u and v.

   (a) Prove that $\mathcal{R}$ is an equivalence relation.

   Solution: For all $u \in V$, $(u, u) \in \mathcal{R}$ by defining of $\mathcal{R}$, implying $\mathcal{R}$ is reflexive. Let $(u, v) \in \mathcal{R}$. Then u and v are connected by a path in G. Since G is undirected, the same path connects v and u and hence $(v, u) \in \mathcal{R}$. The relation is symmetric. Now suppose that $(u, v), (v, w) \in \mathcal{R}$. There are paths $u, u_1, \ldots, u_k, v$ and $v, v_1, \ldots, v_\ell, w$ in G connecting u, v and v, w respectively. Then $u, u_1, u_2, \ldots, u_k, v, v_1, v_2, \ldots, v_\ell, w$ is a path connecting u and w. So $(u, w) \in \mathcal{R}$ thus implying that $\mathcal{R}$ is ttransitive. Clearly $\mathcal{R}$ is an equivalence relation.

   (b) What are the equivalence classes of $V/\mathcal{R}$?

   Solution: Take any connected component of G. Any two vertices in the component are related. Hence the equivalence classes are nothing but the connected components of G.

$\boxed{9+3 = 12}$

4. Prove or disprove the following:

   (a) A group $(G, \cdot)$ is Abelian if and only if $(ab)^2 = a^2b^2$ for all $a, b \in G$.

   Solution: This statement is correct. Suppose that G is Abelian. Then for any $a, b \in G$, we have $(ab)^2 = (ab)(ab) = a(ba)b = a(ab)b = (aa)(bb) = a^2b^2$.

   Now suppose that for all $a, b \in G$, $(ab)^2 = a^2b^2$. We have $abab = aabb$. Multiplying both sides with a^{-1} on the left and b^{-1} on the right, we have: $(a^{-1}a)ba(bb^{-1}) = (a^{-1}a)ab(bb^{-1})$ i.e., $ba = ab$. Therefore G is Abelian.

   (b) Let $(R, +, \cdot)$ be a commutative ring with identity and let 0,1 denoting the identities with respect to $+, \cdot$ respectively. Then, there exists an element in R which is both a unit and a zero-divisor.

   [An element $a \in R \setminus \{0\}$ is called a *unit* if there exists $u \in R$ such that $au = ua = 1$.]

   Solution: This statement is incorrect. Suppose that $a \neq 0$ be both a zero-divisor and a unit in R. Then a has an inverse w.r.t. $\cdot$, say, a^{-1}. Also, since a is a zero-divisor, there exists $b \in R \setminus \{0\}$ such that $ab = 0$. Multiplying both sides with a^{-1} on the left, we have $a^{-1}ab = 0$ i.e., $b = 0$ contradicting our choice of b. Therefore no non-zero a can simultaneously be a zero-divisor and a unit.

$\boxed{6+6 = 12}$

5. Suppose that $A \subseteq \{0, 1\}^*$ is a regular language. Define $\mathsf{DEL\text{-}1}_A = \{xy \mid x1y \in A\}$. The set $\mathsf{DEL\text{-}1}_A$ essentially consists of all strings that can be obtained from strings in A by deleting exactly one 1. Prove that $\mathsf{DEL\text{-}1}_A$ is regular.

   Hint: Use non-determinism. $\boxed{12}$

Solution: Let $\mathcal{M} = (Q, \Sigma = \{0, 1\}, \delta, s, F)$ be a DFA accepting A. We construct a NFA $\mathcal{N} = (Q_\mathcal{N}, \Sigma, \delta_\mathcal{N}, s_\mathcal{N}, F_\mathcal{N})$ for $\mathsf{DEL\text{-}1}_A$. Define $Q_\mathcal{N} = Q \times \{N, Y\}$, $s_\mathcal{N} = (s, N)$, $F_\mathcal{N} = \{(p, Y) \mid p \in F\}$ and the transition function $\delta_\mathcal{N}$ as follows: for every $p \in Q$, $b \in \{0, 1\}$ let

$$\delta_N((p, N), \epsilon) = \{(\delta(p, 1), Y)\}$$
$$\delta_N((p, Y), \epsilon) = \emptyset$$
$$\delta_N((p, N), b) = \{(\delta(p, b), N)\}$$
$$\delta_N((p, Y), b) = \{(\delta(p, b), Y)\}$$

The machine $\mathcal{N}$ basically chooses the 1 to delete non-deterministically. State of the form (p, N) indicates that the 1 has not yet been chosen. There is an ϵ-transition from every such state to state (q, Y) so that $\mathcal{M}$ makes a move from state p to state q upon reading 1. Here, the second component in (q, Y) indicates 1 has been seen and skipped/deleted. This ensures that the computation of $\mathcal{M}$ processing that particular occurrence of 1 is ignored. All other transitions remain as they are in $\mathcal{M}$. Also once a 1 has been deleted, the machine $\mathcal{N}$ behaves exactly as $\mathcal{M}$ i.e., there are no ϵ-transitions from states of the form $(p, 1)$. A string is accepted if after processing it, $\mathcal{N}$ reaches an accepting state after deleting a 1 i.e., it reaches a state of the form (p, Y) where $p \in F$.