Tutorial 8

Time Complexity Classes

1. Prove that the following languages (defined over graphs) are in **P**.
   (a) BIPARTITE – the set of all bipartite graphs. That is, $G = (V, E) \in$ BIPARTITE if V can be partitioned into two sets V_1, V_2 such that every edge in E is adjacent to a vertex in V_1 and a vertex in V_2.
   (b) TRIANGLE-FREE – the set of all graphs that do not contain a triangle (where triangle is a set of three distinct vertices that are mutually connected).

2. Normally, we assume that numbers are represented as strings using the binary basis. That is, a number n is represented by the sequence $x_0, x_1, \ldots, x_{\log n}$ such that $n = \sum_{i=0}^{\log n} x_i 2^i$. However, we could have used other encoding schemes. If $n \in \mathbb{N}$ and $b \geq 2$, then the representation of n in base b, denoted by $\llcorner n \lrcorner_b$ is obtained as follows: first represent n as a sequence of digits in $\{0, \ldots, b-1\}$, and then replace each digit by a sequence of zeroes and ones. The unary representation of n, denoted by $\llcorner n \lrcorner_1$ is the string 1^n (i.e., a sequence of n ones).
   (a) Show that choosing a different base of representation (other than unary) will make no difference to the class **P**. That is, show that for every subset S of the natural numbers, if we define $L_S^b = \{\llcorner n \lrcorner_b : n \in S\}$ then for every $b \geq 2$, $L_S^b \in \mathbf{P}$ iff $L_S^2 \in \mathbf{P}$.
   (b) Show that choosing the unary representation makes a difference by showing that the following language is in **P**.

$$\mathsf{UNARYFACTORING} = \{\langle \llcorner n \lrcorner_1, \llcorner k \lrcorner_1 \rangle : \text{there is a } j \leq k \text{ dividing } n\}.$$

3. Prove that $\mathbf{P} = \mathbf{coP}$ and $\mathbf{P} \subseteq \mathbf{NP} \cap \mathbf{coNP}$.

4. Assuming $\mathbf{NP} \neq \mathbf{coNP}$, show that no **NP**-complete problem can be in **coNP**.

5. Show that the halting problem is **NP**–hard.

6. Let

$$\mathsf{DOUBLESAT} = \{\langle \phi \rangle : \phi \text{ is a CNF formula having at least two satisfying assignments}\}.$$

   Show that DOUBLESAT is **NP**–complete.

7. (a) A *vertex cover* in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that every edge of G is incident on at least one vertex in S. Show that the language

$$\mathsf{VERTEXCOVER} = \{(G, k) \mid \text{graph } G \text{ has a vertex cover of size } \leq k\}$$

   is **NP**-complete.

(b) Let S be a set and let $C = \{X_1, \ldots, X_n\}$ be a collection of n subsets of S (for each $i \in [1, n]$, $X_i \subseteq S$). A set S', with $S' \subseteq S$, is called a hitting set for C if every subset in C contains at least one element in S', i.e., $|X_i \cap S'| \geq 1$ for each $i \in [1, n]$. Let HITSET be the language $\{\langle C, k\rangle : C \text{ has a hitting set of size } k\}$. Prove that HITSET is **NP**–complete.

Example $S = \{a, b, c, d, e, f, g\}$, $C = \{\{a, b, c\}, \{d, a\}, \{d, e, f\}, \{g\}\}$

- $k = 2$, no hitting sets exist.
- $k = 3$, $S' = \{a, d, g\}$ (other choices exist).

Hint: Try reducing from VERTEXCOVER.

8. (Scaling recource bounds.) Let $\mathbf{CL}_1, \mathbf{CL}_2$ denote some time/space complexity classes. Show that if $\mathbf{CL}_1(f(n)) \subseteq \mathbf{CL}_2(g(n))$, then $\mathbf{CL}_1(f(n^c)) \subseteq \mathbf{CL}_2(g(n^c))$.

9. The following two classes are exponential time analogues of **P** and **NP**.

$$\mathbf{EXP} = \cup_{c \geq 1} \mathbf{DTIME}(2^{n^c})$$

$$\mathbf{NEXP} = \cup_{c \geq 1} \mathbf{NTIME}(2^{n^c})$$

Clearly, $\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{EXP} \subseteq \mathbf{NEXP}$. Show that if $\mathbf{EXP} \neq \mathbf{NEXP}$, then $\mathbf{P} \neq \mathbf{NP}$.

Hint: Consider padding strings in **EXP**/**NEXP** languages with exponentially sized strings in order to "scale down" to **P**/**NP**.