1. Do yourself

2. Similar to a problem in Class Test 1.

3. Note that $[a,b) \sim [0,1)$

Define $f : [a,b) \to [0,1)$ as $f(x) = \frac{x-a}{b-a}$

f is well-defined since $b > a$.

Easy to show f is a bijection.

(show it! f^{-1} would be $f^{-1}(y) = (b-a)y + a$)

$\therefore [a,b) \times [c,d) \sim [0,1) \times [0,1)$.

Now let $I = [0,1) \cap \mathbb{Q}$. Since $\mathbb{Q}$ is countable, so are I & I^2.

Consider the sets $X = [0,1)^2 \setminus I^2$ and $Y = [0,1) \setminus I$. Since I, I^2 are countable, we have $X \sim [0,1)^2$ & $Y \sim [0,1)$.

Sufficient to show $X \sim Y$.

Let $x \in X$ and $y \in Y$. Then we can write $x = 0.x_1x_2 \cdots$ & $y = 0.y_1y_2 \cdots$.

Define $g_1 : X \to Y$ as

$g_1(x, y) = 0.x_1 y_1 x_2 y_2 \ldots$

g_1 is 1-1. So Y is atleast as large as X.

Define $g_2 : Y \to X$ as

$g_2(x) = (x, x)$

g_2 is 1-1 too. That is X is atleast as large as X.

$\therefore X \sim Y$.

4. (a) Let $\mathbb{Z}^d[x]$ denote the set of all polynomials of degree d with coefficients from $\mathbb{Z}$.

$\mathbb{Z}[x] = \bigcup_{d=0}^{\infty} \mathbb{Z}^d[x]$

$\mathbb{Z}^d[x]$ is isomorphic to $\underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{d \text{ times}}$ which is countable.

$\mathbb{Z}[x]$: union of countably infinite countable sets & hence countable.

(b) From (a), $\mathbb{Z}[x]$ is countable. Each $f(x) \in \mathbb{Z}[x]$, $f(x) \neq 0$, has finitely many roots & $\therefore$ $\mathbb{A}$ is countable.

5. TRY ON YOUR OWN

6. (a) $\{1\}$: finite bounded subset of $\mathbb{R}$

$\left\{ \frac{1}{2^i} \mid i \in \mathbb{N} \right\} \subseteq [0,1]$

$\hookrightarrow$ countable, bounded below & above by 0, 1 respectively.

(b) $[0,1]$

7. (a) COUNTABLE

Any bounded subset of $\mathbb{Z}$ with lower, upper bounds a, b resp. is a subset of $A = \{a, a+1, \dots, b\}$ $(a \leq b)$.

A is finite $\Rightarrow 2^A$ is finite.
The set of all bounded subsets of $\mathbb{Z}$ is the union of 2^A for all choices of A. Now, there are countably many ways to choose A, i.e., to choose a, b s.t. $a \leq b$. So there are countably many bounded subsets of $\mathbb{Z}$.

(b) UNCOUNTABLE.

Let S = set of all bounded subsets of $\mathbb{Q}$.

We define a map $f : \mathbb{R} \to S$ as follows:

if $r \in \mathbb{Q}$, then $f(r) = \{r\}$.

if $r \in \mathbb{R} \setminus \mathbb{Q}$, then write $r = x.y_1y_2y_3\cdots$

$$f(r) = \{x, x.y_1, x.y_1y_2, x.y_1y_2y_3, \ldots\}$$

$f(r)$ is bounded above by r & below by x. So, $f(r) \in S$.

f is 1-1 & hence S must be at least as large as $\mathbb{R}$. $\therefore S$ is uncountable.

ALGEBRAIC STRUCTURES

1. (a) $h : (\mathbb{N}, +) \to (\mathbb{Z}_4, +_4)$

$h(x) = x \pmod 4$

$h(x_1 + x_2) = (x_1 + x_2) \pmod 4$

$= x_1 \pmod 4 + x_2 \pmod 4$

$= h(x_1) + h(x_2)$

h is a homomorphism.

h is onto (easy to verify).

$\therefore$ h is a monomorphism.

(b) Do yourself.

(c) Define $h : (S_4, \cdot) \to (\mathbb{Z}_4, +_4)$ as follows:

$h(1) = 0$ $\quad$ $h(i) = 1$

$h(-1) = 2$ $\quad$ $h(-i) = 3$

h is 1-1 & onto.

$h(1 \cdot -1) = h(-1) = 2 = 0 + 2 \ (4) = h(1) + h(-1) \ (4)$

$h(1 \cdot i) = h(i) = 1 = 0 + 1 \ (4) = h(1) + h(i) \ (4)$

$h(1 \cdot -i) = h(-i) = 3 = 0 + 3 \ (4) = h(0) + h(3) \ (4)$

$h(-1\cdot i) = h(-i) = 3 = 2+1\ (4) = h(-1)+h(i)\ (4)$
$h(-1\cdot -i) = h(i) = 1 = 2+3\ (4) = h(-1)+h(-i)\ (4)$
$h(i\cdot -i) = h(1) = 0 = 1+3\ (4) = h(i)+h(-i)\ (4)$
[(4) is short for (mod 4)]

Since $h(xy) = h(x)+h(y)\ (4)$ for all $x, y \in S_4$, h is a homomorphism.

Problems 2 - 6 only involve verifying properties of semi-groups / monoids / groups / rings / fields. Do them yourself.