
Tutorial 2

Sets, Relations and Functions

1. Show the following.
 - (a) If $a \in \{\{b\}\}$, then $b \in a$.
 - (b) If $\mathcal{C} = \{\{x\} \mid b \in B\}$, then $\cup_{X \in \mathcal{C}} X = B$.
2. Prove that for all sets A, B the following statements are equivalent:
 - (a) $A \subseteq B$
 - (b) $A \setminus B = \emptyset$.
 - (c) $A \cup B = B$.
 - (d) $A \cap B = A$.
3. For two sets A, B , the *symmetric difference* $A \Delta B$ is defined as $(A \setminus B) \cup (B \setminus A)$. Prove the following for all sets A, B, C .
 - (a) $A \Delta B = (A \cup B) \setminus (A \cap B)$
 - (b) If $A \setminus C = B \setminus C$, then $A \Delta B \subseteq C$.
 - (c) $A \Delta B = \emptyset$ if and only if $A = B$.
4. Let A be the set of all cities in India. Define a binary relation \mathcal{R} on A as follows: for $x, y \in A$, $(x, y) \in \mathcal{R}$ if the distance between x and y is at most 400 km. Determine whether or not \mathcal{R} is reflexive, asymmetric, transitive, anti-symmetric or irreflexive.
5. A relation \mathcal{R} is *circular* if $(x, y), (y, z) \in \mathcal{R}$ implies $(z, x) \in \mathcal{R}$. Prove that \mathcal{R} is an equivalence relation if and only if it is both circular and reflexive.
6. Let A be a set and \mathcal{R} be an equivalence relation on A . Denote by A/\mathcal{R} (read A quotiented by \mathcal{R}) the set of equivalence classes under \mathcal{R} . For $a \in A$, $[a]_{\mathcal{R}}$ denotes the equivalence class to which a belongs.

Let $f : A \rightarrow B$ be a function and let \mathcal{R} be an equivalence relation on B . Define a relation \mathcal{T} on A as follows: for all $a, a' \in A$, $(a, a') \in \mathcal{T}$ if and only if $(f(a), f(a')) \in \mathcal{R}$.

 - (a) Is \mathcal{T} an equivalence relation on A ?
 - (b) Does \mathcal{T} define a partial order on A ?
 - (c) Define a map $\bar{f} : A/\mathcal{T} \rightarrow B/\mathcal{R}$ as $[a]_{\mathcal{T}} = [f(a)]_{\mathcal{R}}$. Show that \bar{f} is well-defined.
 - (d) Prove that \bar{f} is injective.
7. Define a relation \mathcal{R} on $A = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ as $((a, b), (c, d)) \in \mathcal{R}$ if and only if $ad = bc$. Prove that \mathcal{R} is an equivalence relation. Show that A/\mathcal{R} is essentially \mathbb{Q} .
8. Let Σ be an alphabet which is totally ordered i.e., for every $a, b \in \Sigma$, either $a \preceq b$ or $b \preceq a$. Consider the set Σ^* . We define the *lexicographic ordering* \preceq on Σ^* as follows. Let $x = x_1x_2 \dots x_n$ and $y = y_1y_2 \dots y_m$ be two strings in Σ^* with $\{x_i\}_i, \{y_j\}_j \in \Sigma$. We say $x \preceq y$ if: $n < m$ or $n = m$ and there exists an index $i \in \{1, 2, \dots, n\}$ such that $x_j = y_j$ for all $1 \leq j \leq i - 1$ and $x_i \preceq y_i$.
 - (a) Show that (Σ^*, \preceq) is a partial order. Is it a total order?

- (b) Does there exist a least element? What is it?
- (c) What is the greatest element?
- (d) Consider the subset $A = \{x \in \Sigma^* \mid l_1 \leq |x| \leq l_2\}$ where $l_1, l_2 \in \mathbb{Z}^+$ with $l_1 \leq l_2$. What are the minimal/maximal elements of A ? Are there least/greatest elements?
- (e) For the set A defined above, write down 2 different upper bounds and lower bounds. Is there a least upper bound and a greatest lower bound? What are they?
9. Let S be a set. A characteristic function over S is a function $\chi : S \rightarrow \{0, 1\}$. If $A \subseteq S$, then the characteristic function of A is $\chi_A : S \rightarrow \{0, 1\}$ given by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Prove that there exists a bijection between 2^S and the set of all Boolean function on S i.e., $(S \rightarrow \{0, 1\})$.

10. Which of the following are bijections? Justify your answer.

- (a) $f : \mathbb{Z} \rightarrow \mathbb{Z}$, where $f(n) = (-1)^{|n|}n$ for every $n \in \mathbb{Z}$.
- (b) $f : \mathbb{Z} \rightarrow \mathbb{Z}_{19}$ where $f(n) = n \bmod 19$ for every $n \in \mathbb{Z}$.
- (c) $f : \mathbb{R} \rightarrow \mathbb{C}$ where $f(x) = x^3$ for all $x \in \mathbb{R}$.
- (d) $f : \mathbb{N} \rightarrow \mathbb{N}$, where

$$f(n) = \begin{cases} n + 1 & \text{if } n \text{ is even} \\ n - 1 & \text{if } n \text{ is odd} \end{cases}$$