Problem Set 3: The Probabilistic Method

1. A dominating set of an undirected graph $G = (V, E)$ is a set $U \subseteq V$ such that every vertex $v \in V \setminus U$ has atleast one neighbour in U. Suppose that $G = (V, E)$ is a graph with $|V| = n$ and minimum degree $d > 1$. Then show that G has a dominating set of size at most $n\left(\frac{1+\ln(d+1)}{d+1}\right)$.

2. We can generalize the problem of finding a large cut to finding a large k-cut. A k-cut is a partition of the vertices into k disjoint sets, and the value of a cut is the weight of all edges crossing from one of the k sets to another. In class, we considered 2-cuts when all edges had the same weight 1, showing via the probabilistic method that any graph G with m edges has a cut with value at least $m/2$. Generalize this argument to show that any graph G with m edges has a k-cut with value at least $(k-1)m/k$.

3. Use the Lovász Local Lemma to show that, if

$$k\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1$$

   then it is possible to color the edges of K_n with two colors so that it has no monochromatic K_k subgraph. (Here, K_ℓ denotes complete graph on ℓ vertices.)

4. Prove that any graph has a bipartite subgraph containing at least half the total number of edges.

5. A family of subsets $\mathcal{F}$ of $\{1, 2, \ldots, n\}$ is called an *anti-chain* if there is no pair of sets A and B in $\mathcal{F}$ satisfying $A \subset B$.
   (a) Give an example of $\mathcal{F}$ where $|\mathcal{F}| = \binom{n}{\lfloor n/2 \rfloor}$.
   (b) Let f_k be the number of sets in $\mathcal{F}$ with size k. Show that

$$\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \geq 1.$$

   (c) Argue that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$ for any anti-chain $\mathcal{F}$.

6. Let $G = (V, E)$ be a directed graph with minimum outdegree d and maximum indegree D. Prove (using Lovász Local Lemma) that if $k \leq \frac{d}{1+\ln(1+dD)}$, then G contains a directed cycle of length divsible by k.