# Problem Set 2

1. Consider a 3-person game with $S_1 = S_2 = S_3 = \{1, 2, 3, 4\}$. If $u_i(x, y, z) = x + y + z + 4i$ for each $i = 1, 2, 3$, show that the game has a unique Nash equilibrium.

2. Compute all Nash equilibria for the following game for each $a \in (1, \infty)$.

| | A | B |
|---|---|---|
| A | $(a, 0)$ | $(1, 2 - a)$ |
| B | $(1, 1)$ | $(0, 0)$ |

3. Describe a game with a PSNE and an initial strategy profile starting from which best-response dynamics cycles forever.

4. Provide an example of a network congestion game where best-response dynamics takes an exponential number of iterations to converge to a PSNE.

5. In the 2SAT problem, an instance is a boolean formula in conjunctive normal form where every clause has exactly 2 literals (e.g. $(x_1 \vee x_3) \wedge (\neg x_2 \vee x_3) \wedge (x_1 \vee \neg x_4)$). In Local-Weighted-Max-2SAT problem, we are given a set of 2SAT clauses each having a weight. An assignment of the variables is said to satisfy a clause if and only if it makes at least one of its literals true. The goal is to find an assignment which is locally optimal – by changing the assignment of any one variable, it is not possible to increase the sum of weights of the clauses satisfied. Prove that Local-Weighted-Max-2SAT is **PLS**-complete.

6. Consider a 2-player normal form game where Row Player has $m$ actions and Column Player has $n$ actions.
   (a) Derive the maximum number of support pairs $(S_1, S_2)$ of size $|S_1| = |S_2| = k$ that the Support Enumeration Algorithm must evaluate. Summing over all $k \leq \min(m, n)$, find the total number of candidate support pairs.
   (b) For a fixed support pair of size $k$, determine the time complexity to solve the linear system and verify all equilibrium conditions (indifference, non-negativity, and non-deviation).
   (c) Derive the total worst-case running time of the Support Enumeration Algorithm and explain why it is exponential in the input size.

7. Consider a finite exact potential game with potential function $\Phi(s)$. Suppose we define a new game with the identical set of players and strategies, but we modify the utility functions such that the game admits a new potential function $\Phi'(s) = c \cdot \Phi(s) + d$, where $c > 0$ and $d$ are real constants. Prove that the set of Pure Strategy Nash Equilibria in the new game is identical to the set of PSNEs in the original game.

## Algorithmic & Complexity Foundations

1. State Yao's Minimax Principle for randomized algorithms. Use it to prove a lower bound on the expected running time of any randomized algorithm for searching an element in an unsorted array of size $n$.

2. In the context of Polynomial Local Search (**PLS**) and the MAX-CUT problem discussed in class, consider a restricted case. Prove that if the input graph $G = (V, E)$ has a maximum degree of 3, the Local-Max-Cut problem under a single-vertex flip neighborhood can be solved in strongly polynomial time.