Soft Landing Optimal Control using Convex Optimization

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by

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Abstract

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The optimal control of planetary soft landings is a considerable problem in space exploration, necessitating precise landings at designated sites while adhering to stringent fuel and control limitations. We examine the non-convex control bounds and heading limitations encountered in such missions, introducing an innovative approach for their convexification. Our methodology ensures a lossless convex relaxation, facilitating optimum solutions for the original non-convex issue via convex optimization approaches. Through the formulation and resolution of the powered descent guidance problem, we demonstrate that the solution attains fuel-optimal trajectories while adhering to intricate limitations regarding thrust magnitude and direction, an essential need for real-time planetary landing. We enhance and consolidate prior convexification methods, guaranteeing their applicability across diverse restrictions, such as thrust orientation limitations, mass dynamics, and planetary rotational influences.

Contents

Chapter 1

Introduction

1.1 Introduction

Planetary exploration missions have emerged as a pivotal emphasis for scientific and technological progress. An essential component of these missions entails the accurate soft landing of spacecraft on planetary surfaces. The soft landing phase, commonly known as the powered descent, necessitates directing a spacecraft from a high altitude to a specified destination on the planet's surface while optimizing fuel consumption and complying with stringent operating parameters. This process presents a significant problem in optimal control theory due to the intricacies of state and control constraints, non-convexities resulting from thrust limitations, and pointing restrictions.

Traditionally, the optimum control issue for soft landings entails establishing a fuel-optimal thrust profile that adheres to limitations such as thrust magnitude limits and spacecraft states (e.g., location, velocity). The limitations are intrinsically non-convex due to the necessity of maintaining a minimum thrust level, restricting pointing direction, and fulfilling specified terminal requirements (e.g., achieving zero velocity upon landing). The resultant non-convexity complicates the identification of globally optimum solutions by conventional nonlinear optimization methods, particularly given the rigorous demands for onboard real-time calculations.

This study utilizes convexification techniques to convert the original non-convex optimum control issue into a convex optimization problem. Convexification guarantees that resolving the relaxed convex problem produces solutions that are optimum for the original non-convex problem while maintaining feasibility—a characteristic known as lossless convexification. We also integrate thrust aiming limits and other essential mission parameters, including planetary rotational dynamics and real-time computing viability, into the guiding formulation.

Convex optimization techniques have several advantages in this setting, such as polynomial-time convergence assurances, resilience to disturbances, and effective execution via interior-point approaches. Moreover, progress in real-time convex optimization has facilitated onboard calculation, ultimately improving the autonomy and safety of planetary descent and landing operations. Our methodology illustrates the efficacy of utilizing lossless convexification to address the difficulties presented by non-convex restrictions, hence enabling fuel-optimal and precision-controlled soft landing paths.

Chapter 2

Problem Formulation

2.1 Problem Formulation

2.1.1 Original Nonconvex Problem

The planetary soft landing problem can be framed as a finite-horizon optimal control problem that seeks to minimize fuel consumption while guiding a spacecraft to a precise landing target. The problem formulation includes the state and control dynamics, along with various constraints as described below.

2.1.1.1 System Dynamics

The dynamics of the spacecraft are given by:

$$\dot{x}(t) = A(\omega)x(t) + B\left(g + \frac{T_c(t)}{m(t)}\right), \tag{2.1}$$

$$\dot{m}(t) = -\alpha\|T_c(t)\|, \quad t \in [0, t_f], \tag{2.2}$$

where:

- $x(t) = [r(t), \dot{r}(t)]^T \in \mathbb{R}^6$ is the state vector containing position $r(t) \in \mathbb{R}^3$ and velocity $\dot{r}(t) \in \mathbb{R}^3$.
- $T_c(t) \in \mathbb{R}^3$ is the control input representing the thrust vector.
- $m(t)$ is the spacecraft mass, $\alpha > 0$ is the fuel consumption rate constant, and $g \in \mathbb{R}^3$ is the gravitational acceleration vector.
- $A(\omega)$ and B are system matrices that account for the dynamics of the spacecraft with respect to the planetary rotation.

The matrices $A(\omega)$ and B are defined as follows:

$$A(\omega) = \begin{bmatrix} 0 & I \\ -S(\omega)^2 & -2S(\omega) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{2.3}$$

where:

- $S(\omega)$ is the skew-symmetric matrix corresponding to the planet's angular velocity vector $\omega = (\omega_1, \omega_2, \omega_3)^T$, given by:

$$S(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{2.4}$$

2.1.1.2 Objective Function

The objective of the problem is to minimize fuel consumption:

$$\min_{t_f, T_c(\cdot)} \int_0^{t_f} \|T_c(t)\| \, dt. \tag{2.5}$$

2.1.1.3 Constraints

The problem includes several constraints that govern the state and control behavior:

State Constraints

- **Glide Slope Constraint:** The trajectory must lie within a specified cone to ensure a safe approach:

$$\|E(r(t) - r(t_f))\| - c^T(r(t) - r(t_f)) \leq 0, \quad \forall t \in [0, t_f], \tag{2.6}$$

  where $E = \left[e_2^T \, e_3^T\right]^T$ extracts position components, and c defines the cone's geometry.

- **Velocity Constraint:** The spacecraft's velocity must remain below a specified maximum:

$$\|\dot{r}(t)\| \leq V_{\max}, \quad \forall t \in [0, t_f]. \tag{2.7}$$

Control Constraints

- **Thrust Magnitude Bounds:** The thrust is constrained between specified bounds:

$$\rho_1 \leq \|T_c(t)\| \leq \rho_2, \quad \forall t \in [0, t_f]. \tag{2.8}$$

- **Thrust Pointing Constraint:** The thrust direction is limited relative to a reference direction $\hat{n}$:

$$\hat{n}^T \frac{T_c(t)}{\|T_c(t)\|} \geq \cos\theta, \quad \forall t \in [0, t_f]. \tag{2.9}$$

Initial and Terminal Conditions

$$x(0) = x_0, \quad m(0) = m_0, \tag{2.10}$$

$$x(t_f) = x_f, \quad \dot{r}(t_f) = 0. \tag{2.11}$$

2.1.2 Formulation of the Nonconvex Optimal Control Problems

Problem 1 (Nonconvex Minimum Landing Error Problem):

$$\min_{t_f, T_c(\cdot)} \|Er(t_f) - q\|, \tag{2.12}$$

subject to:

$$\dot{x}(t) = A(\omega)x(t) + B\left(g + \frac{T_c(t)}{m(t)}\right), \tag{2.13}$$

$$\dot{m}(t) = -\alpha\|T_c(t)\|, \quad t \in [0, t_f], \tag{2.14}$$

$$x(t) \in X, \quad \forall t \in [0, t_f], \tag{2.15}$$

$$\rho_1 \leq \|T_c(t)\| \leq \rho_2, \quad \hat{n}^T T_c(t) \geq \|T_c(t)\| \cos\theta, \tag{2.16}$$

$$m(0) = m_0, \quad m(t_f) \geq m_0 - m_f > 0, \tag{2.17}$$

$$r(0) = r_0, \quad \dot{r}(0) = \dot{r}_0, \tag{2.18}$$

$$e_1^T r(t_f) = 0, \quad \dot{r}(t_f) = 0. \tag{2.19}$$

Problem 2 (Nonconvex Minimum Fuel Problem):

$$\min_{t_f, T_c(\cdot)} \int_0^{t_f} \alpha\|T_c(t)\| \, dt, \tag{2.20}$$

subject to:

$$\text{dynamics and constraints as in Problem 1,} \tag{2.21}$$

$$\|Er(t_f) - q\| \leq \|d_P^* - q\|, \tag{2.22}$$

where d_P^* denotes the closest reachable point to the target q.

Chapter 3

Convex Relaxation and convex problem formulation

3.1 Convex Relaxation Derivation

The original nonconvex soft landing problem is challenging to solve due to the non-convex constraints on thrust magnitude and pointing direction. To enable efficient computation of globally optimal solutions, we introduce a convex relaxation of these constraints, while guaranteeing that the relaxed problem remains equivalent to the original problem under certain conditions (lossless convexification).

3.1.1 Relaxation of Thrust Constraints

The original control constraints on the thrust magnitude are:

$$\rho_1 \leq \|T_c(t)\| \leq \rho_2, \quad \forall t \in [0, t_f]. \tag{3.1}$$

To convexify this constraint, we introduce a slack variable $\sigma(t)$ such that:

$$\|T_c(t)\| \leq \sigma(t), \quad \rho_1 \leq \sigma(t) \leq \rho_2. \tag{3.2}$$

The slack variable ensures that the feasible set is convex. However, for the relaxed problem to yield an equivalent solution to the original nonconvex problem, the optimal solution must satisfy $\|T_c(t)\| = \sigma(t)$ almost everywhere.

3.1.2 Relaxation of Thrust Pointing Constraint

The original nonconvex constraint on the thrust direction is:

$$\hat{n}^T \frac{T_c(t)}{\|T_c(t)\|} \geq \cos\theta, \quad \forall t \in [0, t_f]. \tag{3.3}$$

By introducing the slack variable $\sigma(t)$ as defined above, this constraint becomes:

$$\hat{n}^T T_c(t) \geq \sigma(t)\cos\theta, \quad \forall t \in [0, t_f], \tag{3.4}$$

which is convex in the variables $T_c(t)$ and $\sigma(t)$.

3.1.3 Modified System Dynamics

The dynamics of the system are also modified to accommodate the slack variable. The original mass consumption dynamics are given by:

$$\dot{m}(t) = -\alpha\|T_c(t)\|. \tag{3.5}$$

In the relaxed problem, this becomes:

$$\dot{m}(t) = -\alpha\sigma(t). \tag{3.6}$$

3.2 Formulation of the Relaxed Convex Optimal Control Problems

With these relaxations, we can now formulate the relaxed convex problems.

Problem 3 (Convex Relaxed Minimum Landing Error Problem):

$$\min_{t_f, T_c(\cdot), \sigma(\cdot)} \|Er(t_f) - q\|, \tag{3.7}$$

subject to:

$$\dot{x}(t) = A(\omega)x(t) + B\left(g + \frac{T_c(t)}{m(t)}\right), \tag{3.8}$$

$$\dot{m}(t) = -\alpha\sigma(t), \quad t \in [0, t_f], \tag{3.9}$$

$$x(t) \in X, \quad \forall t \in [0, t_f], \tag{3.10}$$

$$\|T_c(t)\| \leq \sigma(t), \quad \rho_1 \leq \sigma(t) \leq \rho_2, \tag{3.11}$$

$$\hat{n}^T T_c(t) \geq \sigma(t) \cos\theta, \tag{3.12}$$

$$m(0) = m_0, \quad m(t_f) \geq m_0 - m_f > 0, \tag{3.13}$$

$$r(0) = r_0, \quad \dot{r}(0) = \dot{r}_0, \tag{3.14}$$

$$e_1^T r(t_f) = 0, \quad \dot{r}(t_f) = 0. \tag{3.15}$$

Problem 4 (Convex Relaxed Minimum Fuel Problem):

$$\min_{t_f, T_c(\cdot), \sigma(\cdot)} \int_0^{t_f} \alpha\sigma(t)\, dt, \tag{3.16}$$

subject to:

$$\dot{x}(t) = A(\omega)x(t) + B\left(g + \frac{T_c(t)}{m(t)}\right), \tag{3.17}$$

$$\dot{m}(t) = -\alpha\sigma(t), \quad t \in [0, t_f], \tag{3.18}$$

$$x(t) \in X, \quad \forall t \in [0, t_f], \tag{3.19}$$

$$\|T_c(t)\| \leq \sigma(t), \quad \rho_1 \leq \sigma(t) \leq \rho_2, \tag{3.20}$$

$$\hat{n}^T T_c(t) \geq \sigma(t) \cos\theta, \tag{3.21}$$

$$m(0) = m_0, \quad m(t_f) \geq m_0 - m_f > 0, \tag{3.22}$$

$$r(0) = r_0, \quad \dot{r}(0) = \dot{r}_0, \tag{3.23}$$

$$e_1^T r(t_f) = 0, \quad \dot{r}(t_f) = 0, \tag{3.24}$$

$$\|Er(t_f) - q\| \leq \|d_P^* - q\|. \tag{3.25}$$

The convex relaxation guarantees that the optimal solution to these problems coincides with the solution to the original nonconvex problems under certain conditions, thereby enabling efficient computation using convex optimization techniques.

3.3 Change of Variables and Convex Approximation

One of the primary sources of nonconvexity in the original problem formulation arises from the dynamics of the spacecraft, specifically the term involving the thrust-to-mass ratio $\frac{T_c(t)}{m(t)}$. This term appears in the state dynamics as:

$$\dot{x}(t) = A(\omega)x(t) + B\left(g + \frac{T_c(t)}{m(t)}\right), \tag{3.26}$$

where $m(t)$ is the time-varying mass of the spacecraft, making the dynamics nonlinear and nonconvex.

To address this issue, we introduce a change of variables that linearizes the dynamics while maintaining the feasibility of the solution. The following transformations are applied:

$$\sigma(t) = \frac{\|T_c(t)\|}{m(t)}, \tag{3.27}$$

$$u(t) = \frac{T_c(t)}{m(t)}, \tag{3.28}$$

$$z(t) = \ln(m(t)). \tag{3.29}$$

3.3.1 Transformed Dynamics

With this change of variables, the mass depletion dynamics are reformulated as:

$$\dot{z}(t) = \frac{\dot{m}(t)}{m(t)} = -\alpha\sigma(t), \tag{3.30}$$

where $\alpha > 0$ is the fuel consumption rate constant. The state dynamics now become:

$$\dot{x}(t) = A(\omega)x(t) + B\left(g + u(t)\right). \tag{3.31}$$

This transformation removes the nonlinearity in the term $\frac{T_c(t)}{m(t)}$, resulting in linear state dynamics with respect to the new control variable $u(t)$.

3.3.2 Convex Constraints

In the transformed variables, the original nonconvex constraints on thrust magnitude are replaced with convex constraints on $\sigma(t)$ and $u(t)$. Specifically, the constraints become:

$$\|u(t)\| \leq \sigma(t), \tag{3.32}$$

$$\rho_1 e^{-z(t)} \leq \sigma(t) \leq \rho_2 e^{-z(t)}, \tag{3.33}$$

where ρ_1 and ρ_2 are the minimum and maximum thrust magnitudes, respectively.

To ensure a convex formulation, we approximate the bounds on $\sigma(t)$ using a second-order cone (SOC) constraint:

$$\rho_1 e^{-z_0}\left(1 - (z(t) - z_0) + \frac{(z(t) - z_0)^2}{2}\right) \leq \sigma(t) \leq \rho_2 e^{-z_0}\left(1 - (z(t) - z_0)\right), \tag{3.34}$$

where $z_0(t) = \ln(m_0 - \alpha\rho_2 t)$ is an estimate of the mass trajectory. This approximation ensures that the constraints on $\sigma(t)$ remain convex and compatible with the overall convex optimization framework.

3.3.3 Convex Approximation Analysis

The SOC approximation of the bounds on $\sigma(t)$ introduces a small error relative to the exact constraints, but this error is analytically bounded and shown to be negligible in practical scenarios. The convex formulation ensures that the optimal solution remains feasible and near-optimal for the original nonconvex constraints. Furthermore, by leveraging efficient interior-point methods for second-order cone programming (SOCP), the relaxed problem can be solved in polynomial time, making it suitable for real-time applications in planetary landing missions.

This change of variables and convex approximation transform the original nonconvex problem into a convex problem while preserving essential properties of the solution. Consequently, the optimal control solution obtained using the relaxed problem remains valid and near-optimal for the original problem formulation.

Chapter 4

Numerical Example

4.1 Numerical Example

In this section, we present a numerical example to demonstrate the proposed convex relaxation approach applied to a planetary soft landing problem using specific constraints and initial conditions. This example illustrates the performance of the relaxed formulation under a particular set of conditions.

4.1.1 Problem Setup

The parameters for the numerical example are defined as follows:

- **Time step:** $dt = 1\,\mathrm{s}$
- **Maximum thrust:** $T_{\max} = 24000\,\mathrm{N}$
- **Initial mass:** $m_0 = 2000\,\mathrm{kg}$
- **Final mass constraint:** $m_f = 300\,\mathrm{kg}$

- **Thrust magnitude bounds:** $\rho_1 = 0.2 \times T_{\max}$, $\rho_2 = 0.8 \times T_{\max}$
- **Fuel consumption rate:** $\alpha = 5 \times 10^{-4}\,\text{s/m}$
- **Maximum allowable velocity:** $V_{\max} = 90\,\text{m/s}$
- **Gravity vector on Mars:** $g = (-3.71, 0, 0)^T\,\text{m/s}^2$
- **Planetary angular velocity vector:** $\omega = (2.53\times10^{-5}, 0, 6.62\times10^{-5})^T\,\text{rad/s}$

The initial state of the spacecraft is specified as:

$$r_0 = (2400, 450, -330)\,\text{m},$$
$$\dot{r}_0 = (-40, 45, 0)\,\text{m/s}.$$

The constraints include a glide slope constraint with a minimum angle $\gamma_{gs} = \frac{\pi}{6}$, a thrust pointing constraint with $\theta = \frac{2\pi}{3}$, and a reference direction vector $\hat{n} = (1, 0, 0)$.

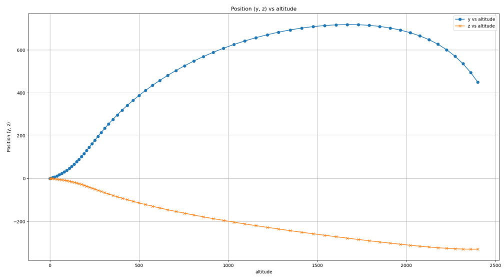
4.1.1.1 Thrust and Position Trajectories

The computed thrust and position trajectories are illustrated in Figure 4.1. Key observations include:

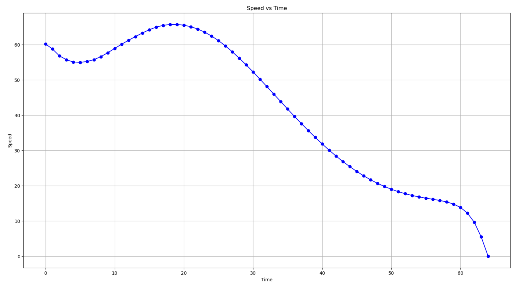
- The thrust magnitude remains within the specified bounds $[\rho_1, \rho_2]$ throughout the descent.
- The trajectory successfully guides the spacecraft to the closest possible spot to the target position, satisfying the glide slope constraint and the thrust pointing constraint.

4.1.1.2 Throttle Profile

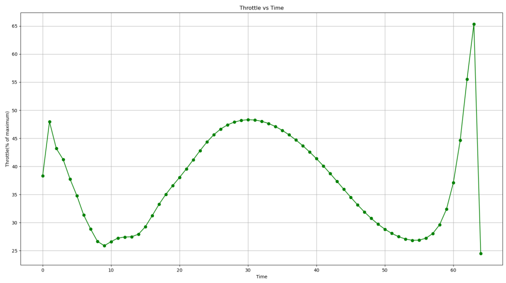
The throttle profile, defined as the normalized thrust magnitude relative to the maximum thrust T_{max}, is shown in Figure 4.1. The throttle remains within the specified bounds and adapts to the constraints on thrust direction.



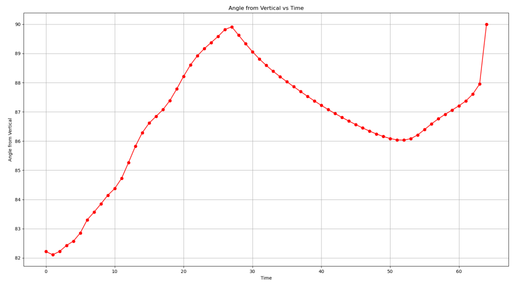
(a) Position v/s Altitude



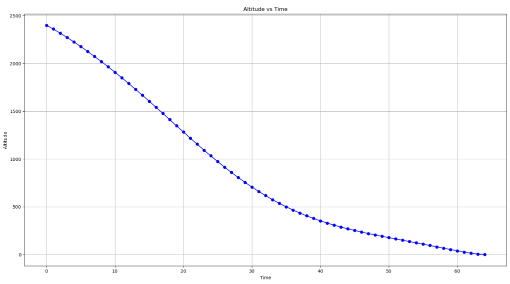
(b) Speed v/s Time



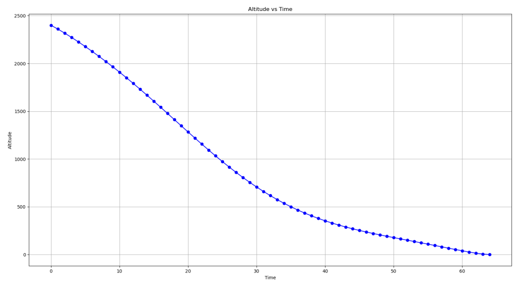
(c) Throttle v/s Time



(d) Angle from vertical v/s Time



(e) Altitude v/s Time



(f) Mass v/s Time

FIGURE 4.1: Various parameters obtained by the algorithm developed

4.1.1.3 Discussion

The results demonstrate the effectiveness of the convex relaxation approach for satisfying all specified constraints while achieving an optimal soft landing trajectory.

As shown, tighter constraints on thrust direction and magnitude lead to adjustments in the trajectory and throttle profile, highlighting the trade-offs inherent in precision landing missions.

4.2 Conclusion

This numerical example illustrates the successful application of convex relaxation techniques for planetary soft landing under specific constraints. The approach offers an efficient method for real-time guidance, ensuring feasible and near-optimal solutions suitable for practical space missions.

Appendix A

A.1 Proof of Equivalence Between Convex Relaxation and Original Nonconvex Problem

In this section, we prove that the convex relaxation described earlier produces the same optimal solution as the original nonconvex problem under specific conditions. This result is often referred to as a lossless convexification, meaning that the solution of the relaxed problem is guaranteed to satisfy the constraints of the original problem while achieving the same optimal value.

A.1.1 Theoretical Background and Lemma

The convex relaxation is based on relaxing the nonconvex constraints on thrust magnitude and pointing direction by introducing a slack variable $\sigma(t)$. We denote the set of feasible solutions for the original nonconvex problem by $\mathcal{F}_{\text{nonconvex}}$ and for the relaxed convex problem by $\mathcal{F}_{\text{convex}}$. The relaxation is considered lossless if:

$$\exists\,(T_c^*(t), \sigma^*(t)) \in \mathcal{F}_{\text{convex}} \text{ such that } \|T_c^*(t)\| = \sigma^*(t) \quad \text{a.e. on } [0, t_f], \tag{A.1}$$

where $(T_c^*(t), \sigma^*(t))$ is an optimal solution of the convex problem.

Lemma A.1. *Given an optimal solution $(T_c^*(t), \sigma^*(t)) \in \mathcal{F}_{convex}$ such that $\|T_c^*(t)\| = \sigma^*(t)$ almost everywhere on $[0, t_f]$, the solution $T_c^*(t)$ also belongs to the feasible set of the original nonconvex problem $\mathcal{F}_{nonconvex}$.*

Proof. Consider the relaxed constraints:

$$\|T_c(t)\| \leq \sigma(t), \quad \rho_1 \leq \sigma(t) \leq \rho_2, \tag{A.2}$$

$$\hat{n}^T T_c(t) \geq \sigma(t) \cos\theta. \tag{A.3}$$

Let $(T_c^*(t), \sigma^*(t))$ be an optimal solution to the relaxed problem. By construction, the slack variable $\sigma(t)$ is introduced to ensure convexity while maintaining a feasible set that contains the feasible region of the original problem. We need to show that:

$$\|T_c^*(t)\| = \sigma^*(t) \quad \text{a.e. on } [0, t_f]. \tag{A.4}$$

Since the cost function is minimized with respect to $\|T_c(t)\|$ (e.g., total fuel consumption), achieving equality $\|T_c^*(t)\| = \sigma^*(t)$ provides the minimum cost for the relaxed problem. If $\|T_c^*(t)\| < \sigma^*(t)$ for any measurable subset of $[0, t_f]$ with positive measure, reducing $\sigma^*(t)$ to match $\|T_c^*(t)\|$ would strictly decrease the cost, contradicting the optimality of $(T_c^*(t), \sigma^*(t))$.

Thus, the condition $\|T_c^*(t)\| = \sigma^*(t)$ holds almost everywhere, implying that the solution $T_c^*(t)$ satisfies the original thrust magnitude constraints:

$$\rho_1 \leq \|T_c^*(t)\| \leq \rho_2. \tag{A.5}$$

Furthermore, the thrust pointing constraint in the relaxed problem:

$$\hat{n}^T T_c^*(t) \geq \sigma^*(t) \cos\theta, \tag{A.6}$$

combined with the condition $\|T_c^*(t)\| = \sigma^*(t)$, ensures that:

$$\hat{n}^T \frac{T_c^*(t)}{\|T_c^*(t)\|} \geq \cos\theta. \tag{A.7}$$

This satisfies the original nonconvex thrust pointing constraint.

Therefore, the optimal solution $T_c^*(t)$ for the relaxed problem is also an optimal solution for the original nonconvex problem. □

A.1.2 Conclusion

This proof establishes that the convex relaxation preserves the optimality and feasibility of solutions for the original nonconvex problem. The approach enables efficient computation through convex optimization while ensuring that the solutions remain valid for the original problem, demonstrating the practical utility of the convexification technique.

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