**CS21201: Discrete Structures**

**Autumn 2025**
**Tutorial 2: Logic**

**Problem 1**
While walking in a labyrinth, you find yourself in front of three possible roads. The road on your left is paved with gold, the road in front of you is paved with marble, while the road on your right is made of small stones. Each road is protected by a guard. You talk to the guards, and this is what they tell you.

• The guard of the gold road: *"This road will bring you straight to the center. Moreover, if the stones take you to the center, then also the marble takes you to the center."*

• The guard of the marble road: *"Neither the gold nor the stones will take you to the center."*

• The guard of the stone road: *"Follow the gold, and you will reach the center. Follow the marble, and you will be lost."*

You know that all the guards are liars. Your goal is to choose a correct road that will lead you to the center of the labyrinth. Solve your problem using a propositional-logic.

**Problem 2**
Your task is to (logically) solve a murder mystery on behalf of Sherlock Holmes, which appeared in the novel "A Study in Scarlet" by Sir Arthur Conan Doyle. The arguments (simplified from the novel) go as follows.

1. There was a murder. If it was not done for robbery, then either it was a political assassination, or it might be for a woman.
2. In case of robbery, usually something is taken.
3. However, nothing was taken from the murderer's place.
4. Political assassins leave the place immediately after their assassination work gets completed.
5. On the contrary, the assassin left his/her tracks all over the murderer's place.
6. For an assassin, to leave tracks all over the murderer's place indicates that (s)he was there all the time (for long duration).

Logically deduce the reason for the murder.

**Problem 3**
Prove the following two logical deductions.

| (a) | (b) |
|---|---|
| $(\neg p \lor q) \to r$ | $t \to q$ |
| $r \to (s \lor t)$ | $\neg r \to \neg s$ |
| $\neg(s \lor u)$ | $p \to u$ |
| $t \to u$ | $\neg t \to \neg r$ |

$q \leftrightarrow v$

$(v \wedge \neg w) \vee (\neg v \wedge w) \rightarrow \neg p$

---

$\therefore \neg w$

$u \rightarrow s$

---

$\therefore p \rightarrow q$

**Question 4**

Someone ate Ronaldo's cookies from his seat in the pavilion. There are three suspects: Messi, Mbappe and Haaland. Ofcourse, all three of them deny stealing. When forced to open their mouths, they say:

Messi: "Ronaldo knows Mbappe, but Haaland liked his cookies."

Mbappe: "I don't even know Ronaldo. Besides, I was on the pitch at that time."

Haaland: "I saw both Messi and Mbappe without Ronaldo in the pavilion at that time, one of them must have done it."

Assume that the two innocent men are telling the truth, but the guilty man may or may not be. Who ate the cookies? Use propositional logic.

**Question 5**

Universal Gates: A universal gate is a boolean logic function that can be used to realize all possible boolean logic functions. NAND and NOR are the two universal gates with their logic as defined below:

$$NAND\ (P,Q) = \neg(P \wedge Q)$$

$$NOR\ (P,Q) = \neg(P \vee Q)$$

Realize the following boolean logic functions using only NAND functions, you may not use the negation ($\neg$) operator:

a. $P \rightarrow Q$

b. $P \leftrightarrow Q$

c. $P \oplus Q$

**Question 6**

Establish the validity of the following statements using the resolution rule (alongwith the rules of inference and the laws of logic):

a. $p \vee (q \wedge r)$

$p \rightarrow s$

---

$\therefore r \vee s$

b. $\neg p \vee s$

$\neg t \vee (s \wedge r)$

$\neg q \vee r$

$p \vee q \vee t$

---

$\therefore r \vee s$

c. $\neg j \vee k$

$k \vee l$

$(l \wedge \neg j) \rightarrow (m \wedge \neg j)$

$\neg k$

---

$\therefore m$

d. $(\neg m \wedge \neg n) \rightarrow (o \rightarrow n)$

$n \rightarrow m$

$\neg m$

---

$\therefore \neg o$

e. $w \rightarrow x$

$(w \wedge x) \rightarrow y$

$(w \wedge y) \rightarrow z$

---

$\therefore w \rightarrow z$

f. $p \rightarrow (q \rightarrow r)$

$p \vee s$

$t \rightarrow q$

$\neg s$

$t \vee u$

---

$\therefore r \vee u$

**Question 7**

As a student of Discrete Structures (DS), consider the following deduction problem to be coded and solved in propositional logic. Use the following propositions specified in the table below:

| Proposition | Meaning |
|---|---|
| S | TRUE, if you study DS. FALSE otherwise |
| D | TRUE, if you do well in the DS exam. FALSE otherwise |
| R | TRUE, if you relax in DS classes. FALSE otherwise |
| E | TRUE, if you enjoy your third semester. FALSE otherwise |

a. Write the following sentences in propositional logic WITHOUT using the implication ($\rightarrow$) operator. You may use any other operator:

S1: If you study DS, then you do well in the DS exam

S2: If you relax in DS classes, then you enjoy your third semester

S3: You either study in DS classes or relax in DS classes, but not both

S4: If you study DS, then you don't enjoy your third semester

S5: If you relax in DS classes, then you don't do well in the DS exam

G: You enjoy your third semester if and only if you do not do well in the DS exam

b. Deduce G from $S1 \wedge S2 \wedge S3 \wedge S4 \wedge S5$ using Truth Tables.

**Question 8**

Show that each of the following arguments is invalid by providing a counterexample – an assignment of truth values of primitive statements p, q, r and s such that all premises are true (have truth value 1) while the conclusion is false (has truth value 0)

(a) $[(p \wedge \neg q) \wedge [p \rightarrow (q \rightarrow r)]] \rightarrow \neg r$

(b) $[[(p \wedge q) \rightarrow r] \wedge (\neg q \vee r)] \rightarrow p$

(c) $p \leftrightarrow q$

$q \rightarrow r$

$r \vee \neg s$

$\neg s \rightarrow \neg q$

$\therefore s$

**(Solutions)**

## Question 1

Let's assign variables:

G: The gold road leads to the center.
M: The marble road leads to the center.
S: The stone road leads to the center.

The guards' statements (all must be FALSE, because they lie):

Gold road guard:

"This road will bring you straight to the center. Moreover, if the stones take you to the center, then also the marble takes you to the center."

Logical form: $G \wedge (S \rightarrow M)$

Since the guard is lying: $\neg[G \wedge (S \rightarrow M)] =$ **$(\neg G) \vee (S \wedge \neg M)$**

Marble road guard:

"Neither the gold nor the stones will take you to the center."

Logical form: $\neg G \wedge \neg S$

Since the guard is lying: $\neg[\neg G \wedge \neg S] =$ **$G \vee S$**

Stone road guard:

"Follow the gold, and you will reach the center. Follow the marble, and you will be lost."

Logical form: $G \wedge \neg M$

Since the guard is lying: $\neg[G \wedge \neg M] =$ **$(\neg G) \vee M$**

We need to find the truth values of G, M, and S that satisfy.Only two combinations satisfy all 3 constraints:

Case 1: G=F, M=F, S=T
Case 2: G= F, M =T, S=T
So in both cases, Stone road (S = T) leads to the center.

## Question 2

Let

R: Murder for robbery.

P: Poltical assassination.

W: committed for women.

T: something was taken from murder's place.

L: the assassin left tracks all over the place.

S: stayed long at the place.

Given:-

1) $\sim R \rightarrow (P \vee W)$

2) $R \rightarrow T$

3) $\sim T$

4) $P \rightarrow \sim S$

5) $L$

6) $L \rightarrow S$

~~from 5 and~~

from $L \rightarrow S$ —— (6)

$L$ —— (5)

$S$ —— (7)

from $P \rightarrow \sim S$ —— (4)

$S$ —— (7)

$\sim P$ —— (8)

from $R \rightarrow T$ —— (2)

$\sim T$ —— (3)

$\sim R$ —— (9)

from $\sim R \rightarrow (P \vee W)$ —— (1)

$\sim R$ —— (9)

$P \vee W$ —— (10)

from $P \vee W$ —— (10)

$\sim P$ —— (8)

$W$ $\Rightarrow$ the murder was committed for a women.

## Question 3

**(a)** Given:

(1) $(\neg p \vee q) \rightarrow r$

(2) $r \rightarrow (s \vee t)$

(3) $\neg(s \vee u)$

(4) $t \rightarrow u$

(5) $q \leftrightarrow v$
(6) $(v \land \neg w) \lor (\neg v \land w) \to \neg p$

From (3) using De Morgan’s Law:
$\Rightarrow \neg s \land \neg u$
$\Rightarrow$ So:
(7) $\neg s$
(8) $\neg u$

From (4): $t \to u$
Contrapositive: $\neg u \to \neg t$
From (8) and contraposition:
(9) $\neg t$

From (2): $r \to (s \lor t)$
Contrapositive: $\neg(s \lor t) \to \neg r$
From (7) and (9): $\neg s \land \neg t \Rightarrow \neg(s \lor t)$
$\Rightarrow$ (10) $\neg r$

From (1): $(\neg p \lor q) \to r$
Contrapositive: $\neg r \to \neg(\neg p \lor q)$
From (10):
$\Rightarrow \neg(\neg p \lor q)$
Apply De Morgan’s Law again:
$\neg(\neg p \lor q) \equiv p \land \neg q$
$\Rightarrow$ (11) $p$
$\Rightarrow$ (12) $\neg q$

From (5): $q \leftrightarrow v$
So $\neg q \leftrightarrow \neg v$
From (12):
$\Rightarrow$ (13) $\neg v$

Assume $w$ = true:
Then $(\neg v \land w)$ is true (since $\neg v$ is true from (13))
$\Rightarrow$ LHS of (6) is true
$\Rightarrow$ RHS must also be true $\Rightarrow \neg p$
But from (11): $p$
So contradiction!
Therefore assumption is false
$\Rightarrow$ (14) $\neg w$

∴ Final Answer for (a): ¬w

**(b)** Given:

(1) t → q
(2) ¬r → ¬s
(3) p → u
(4) ¬t → ¬r
(5) u → s

Assume: p [Assumption for conditional proof]
From (3):
⇒ u

From (5):
u → s
⇒ s

From (2):
¬r → ¬s
Contrapositive: s → r
From s:
⇒ r

From (4):
¬t → ¬r
Contrapositive: r → t
From r:
⇒ t

From (1):
t → q
From t:
⇒ q

Thus, from p ⇒ q
∴ Final Answer for (b): p → q

**<u>Question 4</u>**

**Solution I**: Assume that Ronaldo knows Mbappe means that they both know each other. Propositions used are as follows:

K : Mbappe knows Ronaldo (Ronaldo knows Mbappe)
HL : Haaland likes the cookies
MBP : Mbappe was on the pitch
MEP : Messi was on the pitch

Messi: $K \wedge HL$
Mbappe: $\neg K \wedge \neg MBP$
Haaland: $MBP \wedge MEP$

Since one and only one of Haaland, Mbappe or Messi ate the cookies. We break the problem down into three cases depending on who ate the cookies:

a. <u>Messi:</u> Both Mbappe and Haaland are telling the truth. But for that to be true, $\neg MBP$ and $MBP$ have to be true at the same time, **not possible**
b. <u>Haaland:</u> Both Messi and Mbappe are telling the truth. But for that to be true, $K$ and $\neg K$ have to be true, **not possible**
c. <u>Mbappe:</u> It can be seen that this is the only possible case. Since we can keep HL = 1 and MEP = 1.

**Solution II**: Assume that Ronaldo knows Mbappe but Mbappe does not know Ronaldo. In that case, we design two predicates:
KMR : Mbappe knows Ronaldo
KRM : Ronaldo knows Mbappe
Messi: $KRM \wedge HL$
Mbappe: $\neg KMR \wedge \neg MBP$
Haaland: $MBP \wedge MEP$

Since one and only one of Haaland, Mbappe or Messi ate the cookies. We break the problem down into three cases depending on who ate the cookies:

a. <u>Messi:</u> Not possible, see **Solution I**
b. <u>Haaland:</u> Both Messi and Mbappe are telling the truth. But for that to be true, $KRM$ and $\neg KMR$ have to be true (in addition to $HL$ and $\neg MBP$), **possible**
c. <u>Mbappe:</u> Possible, see **Solution I**

Under this assumption, the identity of the thief cannot be ascertained.

**Question 5**

Denote the $NAND(x)$ function as $N(x)$. You can write $\neg X$ as $N(X, X)$.

a. $P \rightarrow Q \equiv \neg P \vee Q \equiv \neg(P \wedge \neg Q)$

$\equiv N(P, \neg Q)$ (Using De Morgan's Theorem)

$\equiv N(P, N(Q, Q))$

b. $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$

$\equiv N(P, N(Q, Q)) \wedge N(Q, N(P, P))$

To simplify things a bit, choose:
$X = N(P, N(Q, Q))$ and $Y = N(Q, N(P, P))$

$P \leftrightarrow Q \equiv X \wedge Y \equiv \neg(\neg X \vee \neg Y)$ (using De Morgan's Theorem)

$\equiv N(N(X, X), N(Y, Y))$

c. $P \oplus Q \equiv (P \vee Q) \wedge (\neg Q \vee \neg P)$

$X = (P \vee Q) \equiv N(\neg P, \neg Q) \equiv N(N(P, P), N(Q, Q))$

$Y = (\neg P \vee \neg Q) \equiv N(P, Q)$

Repeat the solution to part (b).

**Question 6**

Q3

a. $p \vee (q \wedge r)$

$p \to s \equiv \neg p \vee s$

---

$\therefore (q \wedge r) \vee s \equiv (q \vee s) \wedge (r \vee s)$

$\therefore r \vee s$ [If multiple conditions are joined by $\wedge$, each are individually concluded true]

(proved)

b. $\neg p \vee s$

$p \vee q \vee t$

---

$\therefore s \vee q \vee t$

$s \vee q \vee t$

$\neg t \vee (s \wedge r)$

---

$\therefore s \vee q \vee (s \wedge r)$

$\equiv (s \vee q) \wedge (s \vee q \vee r)$

$s \vee q$

$\neg q \vee r$

---

$\therefore r \vee s$ (proved)

c. $\neg j \vee k$

$\neg k$

---

$\therefore \neg j$

$k \vee l$

$\neg k$

---

$\therefore l$

Since $\neg j$ and $l$ are both true,

$\neg j, \neg l$

$(l \wedge \neg j) \to (m \wedge \neg j)$

---

$\therefore m, \neg j$

(proved)

d. $x \to m \equiv \neg x \vee m$

$\neg m$

---

$\therefore \neg x$

$\neg m, \neg x$

$(\neg m \wedge \neg x) \to (o \to x)$

---

$\therefore o \to x \equiv \neg o \vee x$

$\neg o \vee x$

$\neg x$

---

$\therefore \neg o$ (proved)

e. $w \to x \equiv \neg w \vee x$

$(w \wedge x) \to y \equiv \neg w \vee \neg x \vee y$

---

$\therefore \neg w \vee y$

$\neg w \vee y$

$(w \wedge y) \to z \equiv \neg w \vee \neg y \vee z$

---

$\therefore \neg w \vee z \equiv w \to z$ (proved)

f. $p \vee s$

$\neg s$

---

$\therefore p$

$t \to q \equiv \neg t \vee q$

$t \vee u$

---

$\therefore q \vee u$

$p$

$p \to (q \to r) \equiv \neg p \vee (q \to r)$

---

$\therefore q \to r \equiv \neg q \vee r$

$q \vee u$

$\neg q \vee r$

---

$\therefore u \vee r$ (proved)

**Question 7**

Coding using Propositional Logic is as follows:

S1 : $\neg S \vee D$

S2 : $\neg R \vee E$

S3 : $(S \wedge \neg R) \vee (\neg S \wedge R)$

S4 : $\neg S \vee E$

S5 : $\neg R \vee D$

G : $(\neg E \vee \neg D) \wedge (D \vee E)$

The goal should be easy enough to derive using the truth table method

**Question 8**

Q5

(a)

| p | q | r | $p \wedge \neg q$ | $p \to q \to r$<br>$\neg p \vee \neg q \vee r$ | $\neg r$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 | 1 | 0 |

$(p \wedge \neg q)$ and $[p \to (q \to r)]$ both are true, but $\neg r$ is false.

(b)

| p | q | r | $(p \wedge q) \to r$<br>$\neg p \vee \neg q \vee r$ | $\neg q \vee r$ | p |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

Multiple counter-examples.

(c)

| p | q | r | s | $p \leftrightarrow q$ | $q \to r$ | $s \to r$ ($\equiv r \vee \neg s$) | $\neg s \to \neg q$ ($\equiv s \vee \neg q$) | s |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Counter-examples [rows 1 and 3 boxed]