Tutorial 4
Discrete Mathematics

Practice Solutions

Countable and Uncountable Sets

1. Let A and B be uncountable sets with $A \subseteq B$. Prove or disprove: A and B are equinumerous.

   Solution:

   The statement is **false**. An uncountable subset of an uncountable set need not be equinumerous with the larger set.

   Counterexample: Let $B = \mathcal{P}(\mathbb{R})$ and $A = \{\{x\} \mid x \in \mathbb{R}\}$.

   (a) $A \subseteq B$: each $\{x\}$ is a subset of $\mathbb{R}$, hence an element of $\mathcal{P}(\mathbb{R})$.
   (b) A is uncountable: the map $f : \mathbb{R} \to A$, $f(x) = \{x\}$ is a bijection, so $|A| = |\mathbb{R}| = \mathfrak{c}$.
   (c) B is uncountable: $|B| = |\mathcal{P}(\mathbb{R})| = 2^{\mathfrak{c}}$.

   Comparison:

   $$|A| = \mathfrak{c}, \qquad |B| = 2^{\mathfrak{c}}, \qquad \mathfrak{c} < 2^{\mathfrak{c}} \text{ (Cantor's Theorem).}$$

   Thus $|A| \neq |B|$, so A and B are not equinumerous.

2. Let A be an uncountable set and B a countably infinite subset of A. Prove or disprove: A is equinumerous with $A \setminus B$.

   Solution:

   The statement is **true**. If A is uncountable and $B \subseteq A$ is countably infinite, then A is equinumerous with $A \setminus B$.

   Construction: Since A is uncountable and B is countable, $A \setminus B$ is uncountable. In particular, it contains a countably infinite subset C. Write

   $$B = \{b_1, b_2, b_3, \dots\}, \qquad C = \{c_1, c_2, c_3, \dots\} \subseteq A \setminus B.$$

   Define $f : A \to A \setminus B$ by

   $$f(b_n) = c_{2n-1}, \qquad f(c_n) = c_{2n}, \qquad f(x) = x \text{ for } x \in A \setminus (B \cup C).$$

   Verification:

   (a) **Codomain:** Every image lies in $A \setminus B$, since no b_n remains in the image.
   (b) **Injective:** Images of B, odd-indexed C, and $A \setminus (B \cup C)$ are disjoint, so no collisions occur.
   (c) **Surjective:**
      - If $y \in (A \setminus B) \setminus C$, then $f(y) = y$.
      - If $y = c_{2n}$, then $f(c_n) = y$.
      - If $y = c_{2n-1}$, then $f(b_n) = y$.

      Thus every element of $A \setminus B$ is attained.

   Conclusion: The function f is a bijection, hence

   $$A \cong A \setminus B.$$

   (Equivalently: for any infinite cardinal κ, $\kappa + \aleph_0 = \kappa$.)

3. Prove that the real interval $[0, 1)$ is equinumerous with the unit square $[0, 1) \times [0, 1)$.

   Solution The sets $\mathbb{F} = \mathbb{Q} \cap [0, 1)$ and $\mathbb{F}^2$ are countable. Therefore $A = [0, 1) - \mathbb{F}$ and $B = [0, 1)^2 - \mathbb{F}^2$ are equinumerous with $[0, 1)$ and $[0, 1)^2$, respectively.

   Now, define the map

   $$f : B \to A, \quad (0.a_1a_2a_3 \ldots, 0.b_1b_2b_3 \ldots) \mapsto 0.a_1b_1a_2b_2a_3b_3 \ldots.$$

   Clearly, f is injective. Thus, $|B| \leq |A|$.

   The other inequality $|A| \leq |B|$ is simpler: map

   $$0.c_1c_2c_3 \ldots \quad \mapsto \quad (0.c_1c_2c_3 \ldots, 0.c_1c_2c_3 \ldots).$$

4. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Show that, $[a, b) \times [c, d)$ is equinumerous with $[0, 1)$.

   Solution:

   Define $f : [a, b) \to [0, 1)$ such that

   $$f(x) = \frac{x - a}{b - a}$$

   Define $g : [c, d) \to [0, 1)$ such that

   $$g(y) = \frac{y - c}{d - c}$$

   Now, show that f is bijective (show f is one to one and onto). Similarly, g is also bijective.

   Define $h : [a, b) \times [c, d] \to [0, 1)^2$ such that

   $$h(x, y) = (f(x), g(y))$$

   Now, show that h is bijective (show h is one to one and onto). We have shown that $[a, b) \times [c, d)$ is equinumerous with $[0, 1)^2$.

   From 3rd question, we know $[0, 1)^2$ is equinumerous with $[0, 1)$

5. Define a relation $\sim$ on $\mathbb{R}$ such that $a \sim b$ if and only if $a - b \in \mathbb{Q}$. Answer the following:

   (a) Prove that $\sim$ is an equivalence relation.

   (b) Is the set $\mathbb{R}/\sim$ of all equivalence classes of $\sim$ countable?

   Solution:

   (a) We show that $\sim$ is an equivalence relation on $\mathbb{R}$.

   - Reflexive: For all $a \in \mathbb{R}$, $a - a = 0 \in \mathbb{Q}$. Hence $a \sim a$. - Symmetric: If $a \sim b$, then $a - b \in \mathbb{Q}$. Thus $b - a = -(a - b) \in \mathbb{Q}$, so $b \sim a$. - Transitive: If $a \sim b$ and $b \sim c$, then $a - b \in \mathbb{Q}$ and $b - c \in \mathbb{Q}$. Adding gives $a - c = (a - b) + (b - c) \in \mathbb{Q}$, so $a \sim c$.

   Therefore $\sim$ is an equivalence relation.

   (b) Consider the set $\mathbb{R}/\sim$ of equivalence classes.

   For $x \in \mathbb{R}$,

   $$[x] = \{ y \in \mathbb{R} : y - x \in \mathbb{Q} \} = \{ x + q : q \in \mathbb{Q} \}.$$

   The map $q \mapsto x + q$ is a bijection $\mathbb{Q} \to [x]$, so each equivalence class $[x]$ is countable.

   The distinct equivalence classes correspond to distinct cosets of $\mathbb{Q}$ in $\mathbb{R}$. If $[x] = [y]$, then $x - y \in \mathbb{Q}$; if $[x] \neq [y]$, then $x - y \notin \mathbb{Q}$. Thus $\mathbb{R}/\sim$ has as many elements as $\mathbb{R}/\mathbb{Q}$, which is uncountable.

   Hence each class is countable, but the set of all equivalence classes $\mathbb{R}/\sim$ is uncountable.

6. Let $\mathbb{Z}[x]$ denote the set of all univariate polynomials with integer coefficients.

   Answer the following:

   (a) Prove that $\mathbb{Z}[x]$ is countable.

   Solution $\mathbb{Z}[x]$ is the countable union of $\{0\}$ and $\mathbb{Z}_d[x]$ for $d \in \mathbb{N}_0$, where $\mathbb{Z}_d[x]$ is the set of all univariate polynomials with integer coefficients and degree exactly equal to d. Such a polynomial can be written as

   $$a_d x^d + a_{d-1}x^{d-1} + \cdots + a_2x^2 + a_1x + a_0,$$

   with $a_i \in \mathbb{Z}$ and $a_d \neq 0$. Since each a_i has countably many possibilities, and there are only finitely many coefficients ($d+1$ of them), each $\mathbb{Z}_d[x]$ is countable.

   (b) A real or complex number a is called algebraic if $f(a) = 0$ for some non-zero $f(x) \in \mathbb{Z}[x]$. Let $\mathbb{A}$ denote the set of all algebraic numbers. Prove that $\mathbb{A}$ is countable.

   Solution There are countably many polynomials in $\mathbb{Z}[x] \setminus \{0\}$. Each such polynomial has only finitely many roots.

   (c) Prove that there are uncountably many transcendental (i.e. non-algebraic) numbers.

   Solution $\mathbb{R}$ is the disjoint union of $\mathbb{R} \cap A$ and the set $\mathbb{T}$ of all (real) transcendental numbers. Since A is countable, so too is $\mathbb{R} \cap A$. If $\mathbb{T}$ is countable, then $\mathbb{R}$ is countable too.

7. Let $\mathbb{Z}[x, y]$ be the set of all bivariate polynomials with integer coefficients.

   Answer the following:

   (a) Prove that $\mathbb{Z}[x, y]$ is countable.

   (b) Let $V = \big\{(a, b) \in \mathbb{C} \times \mathbb{C} \mid f(a, b) = 0 \text{ for some nonzero } f(x, y) \in \mathbb{Z}[x, y]\big\}$. Is V countable?

   Solution:

   (a) <u>$\mathbb{Z}[x, y]$ is countable.</u>

   Any polynomial $f(x, y) \in \mathbb{Z}[x, y]$ can be written as a finite sum of monomials

   $$f(x, y) = \sum a_{ij} x^i y^j,$$

   where $a_{ij} \in \mathbb{Z}$ and $i, j \in \mathbb{N}_0$. Each polynomial is thus uniquely represented by a finite set of triples $(a_{ij}, i, j) \in \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0$.

   1. The set $A = \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0$ is countable since it is a finite product of countable sets. 2. A polynomial corresponds to a finite sequence of elements of A. The collection of all finite sequences from A can be written as

   $$S = \bigcup_{k=1}^{\infty} A^k,$$

   where A^k is the set of k-tuples from A. Each A^k is countable, and the countable union of countable sets is countable. 3. Thus, $\mathbb{Z}[x, y]$ injects into S, so $\mathbb{Z}[x, y]$ is countable.

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   (b) <u>V is uncountable.</u>

   Define

   $$V = \{(a, b) \in \mathbb{C} \times \mathbb{C} : \; f(a, b) = 0 \text{ for some nonzero } f(x, y) \in \mathbb{Z}[x, y]\}.$$

   1. Since $\mathbb{Z}[x, y]$ is countable, we may enumerate its nonzero elements as $\{f_k : k \in \mathbb{N}\}$. For each f_k, define its zero set

   $$V_k = \{(a, b) \in \mathbb{C}^2 : \; f_k(a, b) = 0\}.$$

Then

$$V = \bigcup_{k=1}^{\infty} V_k.$$

2. Consider $f(x, y) = x - c$ for some fixed $c \in \mathbb{Z}$. Its zero set is

$$V_c = \{(a, b) \in \mathbb{C}^2 : a - c = 0\} = \{(c, b) : b \in \mathbb{C}\}.$$

This set is in bijection with $\mathbb{C}$, which is uncountable.

3. Since V contains V_c as a subset, and V_c is uncountable, it follows that V is uncountable.

$\square$

8. A set $S \subseteq \mathbb{R}$ is called bounded if S has both a lower bound and an upper bound.

   Provide examples for the following.

   (a) Countable bounded subset of $\mathbb{R}$.

   (b) Uncountable bounded subset of $\mathbb{R}$.

   Determine whether the following sets are countable/uncountable?

   (c) The set of all bounded subsets of $\mathbb{Z}$.

   (d) The set of all bounded subsets of $\mathbb{Q}$.

   Solution:

   (a) A countable bounded subset of $\mathbb{R}$: Any finite subset of $\mathbb{R}$, is countable and will have a minimum and maximum.

   (b) An uncountable bounded subset of $\mathbb{R}$:

   $$S = [0, 1].$$

   This set is bounded below by 0 and above by 1. It is uncountable by Cantor's diagonal argument.

   (c) The set of all bounded subsets of $\mathbb{Z}$ is countable.

   A bounded subset of $\mathbb{Z}$ means a set of integers that lies between some lower bound l and upper bound u. If $S \subseteq \mathbb{Z}$ has bounds $l, u \in \mathbb{R}$, then

   $$S \subseteq [\lceil l \rceil,\ \lfloor u \rfloor] \cap \mathbb{Z}.$$

   Hence S is contained in some finite interval of integers.

   The power set of a finite set is finite (hence countable). Since there are only countably many choices for the integer bounds l, u, the collection of all bounded subsets of $\mathbb{Z}$ is a countable union of finite sets, which is countable.

   Therefore, the set of all bounded subsets of $\mathbb{Z}$ is countable.

   (d) The set of all bounded of $\mathbb{Q}$.

   Uncountable. Let B denote the set of all bounded subsets of $\mathbb{Q}$. Define $f : \mathbb{R} \to B$ as follows: - If $x \in \mathbb{Q}$, set $f(x) = \{x\}$. - If $x \in \mathbb{R} \setminus \mathbb{Q}$, let $a = \lfloor x \rfloor$. Then $x - a \in [0, 1)$ has infinite decimal expansion $0.d_1 d_2 d_3 \ldots$. Define

   $$f(x) = \{a,\ a + 0.d_1,\ a + 0.d_1 d_2,\ a + 0.d_1 d_2 d_3,\ \ldots\}.$$

   Each $f(x)$ is a bounded subset of $\mathbb{Q}$, and f is injective. Thus B is uncountable.

9. Provide a diagonalization argument to prove that the set of all infinite bit sequences is uncountable.

   Solution:

   Assume S (all infinite bit sequences) is countable. Then there is a bijection

   $$f : \mathbb{N} \to S.$$

   Write

   $$\begin{aligned} f(1) &= a_{11}, a_{12}, a_{13}, \dots \\ f(2) &= a_{21}, a_{22}, a_{23}, \dots \\ f(3) &= a_{31}, a_{32}, a_{33}, \dots \\ &\vdots \\ f(n) &= a_{n1}, a_{n2}, a_{n3}, \dots \\ &\vdots \end{aligned}$$

   Define the diagonal-complement sequence

   $$s = (a'_{11}, a'_{22}, a'_{33}, \dots, a'_{nn}, \dots),$$

   where a'_{ij} is the bit complement of a_{ij}.

   For each n, the n-th bit of s is $a'_{nn} \neq a_{nn}$, so $s \neq f(n)$ for all n. Hence f is not surjective, a contradiction. Therefore S is uncountable.

1.

Consider the set $S = \{a + b\sqrt{7} \mid a, b \in \mathbb{Z}\}$. Prove that $\mathbb{R} - S$ is uncountable.

Note that S is countable, since $f : \mathbb{Z} \times \mathbb{Z} \to S$ is a bijection. Assume that $\mathbb{R} - S$ is countable, then $(\mathbb{R} - S) \cup S = \mathbb{R}$ is countable, since it will be the union of two countable sets. However, $\mathbb{R}$ is uncountable and therefore our assumption is wrong.

By the same logic, for any countable set $S, \mathbb{R} - S$ is uncountable.

2.

Provide an explicit bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. It should not be an exhaustive enumeration.

Think about the exhaustive enumeration method where you tried to enumerate all the coordinates in a diagonal scheme.
For a natural number N, try to find out how many diagonals can be completely enumerated.

First diagonal enumerates (1, 1) : 1 point
Second diagonal enumerates (2, 1) and (1, 2) : 2 points

Third diagonal enumerates (3, 1), (2, 2) and (1, 3) : 3 points
nth diagonal enumerates (n, 1), ($n - 1$, 2) …. (1, n) : n points

This is the maximum n such that $\frac{n(n+1)}{2} \leq N$. Find this n. Let $d = N - n$
$[d = 0]$ The mapping is given by $N \to (1, n)$
$[0 < d \leq n + 1]$ The mapping is given by $N \to \big(n + 1 - (d - 1), 1 + (d - 1)\big) = (n - d + 2, d)$

3.

Determine whether the following sets are countable or uncountable:

a) The set of all finite subsets of $\mathbb{N}$

Countable

Let this set be A. We find a way to enumerate all the finite subsets of $\mathbb{N}$ using the following enumeration scheme

{}
{1} {2} {3} {4} {5} {6}
{1, 2} {1, 3} {1, 4} {1, 5} {1, 6} {1, 7}
{1, 2, 3} {1, 2, 4} {1, 2, 5} {1, 2, 6} {1, 2, 7} …
{1, 2, 3, 4} {1, 2, 3, 5} {1, 2, 3, 6} {1, 2, 3, 7} {1, 2, 3, 8} …
{1, 2, 3, 4, 5} {1, 2, 3, 4, 6} {1, 2, 3, 4, 7} {1, 2, 3, 4, 8} {1, 2, 3, 4, 9} …

Alternate Method: We also know that a countable union of countable sets is countable. The above set can be written as $S = S_0 \cup S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_n \cup \ldots$ where $S_i \subseteq \mathbb{N}^i$. Each of $\mathbb{N}^i$ is countable.

b) The set of all infinite subsets of $\mathbb{N}$

Uncountable

Let this set be B. Assume that B is countable, then $A \cup B$ is also countable, since it is a union of a countable number of countable sets. But $A \cup B$ is the number of subsets of $\mathbb{N}$ which is obviously uncountable. Therefore, our assumption is wrong and B is uncountable.

4.

Infinite Bit Sequences] As the name suggests, an infinite bit sequence is an infinite sequence of 0s and 1s. Denote S as the set of all infinite bit sequences. Let $\alpha(n)$ be the nth element of an infinite bit sequence $\alpha \in S$. Determine whether the following sets are countable or uncountable:

a) S

Uncountable

A simple diagonalization argument would suffice. Construct an infinite bite sequence β such that $\beta(n) = \overline{\alpha_n(n)}$, where α_n denotes the α which is mapped to integer $n \in \mathbb{N}$. Clearly, $\beta \in S$. Let $\beta = \alpha_k$ for some $k \in \mathbb{N}$. But $\beta(k) \neq \alpha_k(k)$ by virtue of the above construction. Hence S is uncountable

b) $T_1 = \{\alpha \in S \mid \alpha(n) = 1 \text{ and } \alpha(n+1) = 0 \text{ for } \underline{\text{some}}\ n \geq 0\}$

Uncountable

Consider the set $T_3 = \{\alpha \in S \mid \alpha(0) = 1 \text{ and } \alpha(1) = 0\}$. We have $T3 \subseteq T1$, and so $|T_3| \leq |T_1| \leq |S|$ (use the canonical inclusion maps which are injective).
On the other hand, take any $\alpha = (1, 0, a_2, a_3, \ldots, a_n, \ldots,) \in T_3$. The map taking $\alpha \rightarrow (a_2, a_3, a_4, \ldots, a_{n+2}, \ldots) \in S$ is clearly a bijection $T_3 \rightarrow S$, implying that $|T_3| = |S|$.

c) $T_2 = \{\alpha \in S \mid \alpha(n) = 1 \text{ and } \alpha(n+1) = 0 \text{ for } \underline{\text{no}}\ n \geq 0\}$

Countable

Each sequence of T_2 is of the format 0000….00001111… or 0000…
Consider the bijective map $f : T_2 \rightarrow \mathbb{N}$ such that $f(0000 \ldots\ldots) = 1$

and for all other sequences $\alpha \in T_2, f(\alpha) = n + 2$, where n is the number of zeros.

5.

[Sets of Functions] We have the following sets, determine whether they are countable or uncountable.

a) The set of all functions from $\mathbb{N}$ to $\{1, 2\}$

Uncountable

Assume that this set (S) is countable. Therefore, there is a bijective mapping $f : \mathbb{N} \rightarrow S$ such that $a \in \mathbb{N} \rightarrow f_a \in S$. Consider the following diagonalization argument:

	f_1	f_2	f_3	f_4	..	..	f_n	..	..
1	**1**	2	2	1	..	..	2	..	..
2	1	**1**	2	1	..	..	1	..	..
3	2	2	**1**	2	..	..	2	..	..
4	1	1	1	**2**	..	..	1	..	..
..	..	..	..	..	..	..	..	..	..
..									
n	2	1	2	2	..	..	**1**	..	..
..									

Consider a function F which gives outputs opposite of those given by f_i across the diagonal above.

$$F(x) = \begin{cases} 1 & , f_x(x) = 2 \\ 2 & , f_x(x) = 1 \end{cases} \quad \forall x \in \mathbb{N}$$

$F \in S$. However, this means that $F = f_k$ for some k. But, if $f_k(k) = 1$, then $F(k) = 2$. Hence our assumption is invalid.

b) The set of all functions from $\mathbb{N}$ to $\mathbb{N}$

Uncountable, any superset of an uncountable set is also uncountable. Alternatively, the above diagonalization argument can be modified to prove this statement by selecting $F(x) = f_x(x) + 1$.

c) The set of all functions from $\{1, 2\}$ to $\mathbb{N}$

Countable, notice that this set will have the same cardinality as $\mathbb{N} \times \mathbb{N}$. Let this set be S. Each $f \in S$ has two values $f(1)$ and $f(2)$, both of which are in $\mathbb{N}$.

d) The set of all non-increasing functions from $\mathbb{N}$ to $\mathbb{N}$
Countable, this set is the countable union of countable sets. Let this set be denoted by S. Let $f \in S$. Then:

$$f(1) \geq f(2) \geq f(3) \geq \cdots$$

Eventually, $\forall f \in S$, there exist values i and n_0 such that $\forall n \geq n_o[f(n) = i]$.
Mathematically,

$$\exists n_0 \exists i \ (\forall n \geq n_o)[f(n) = i]$$

For a fixed (n_o, i) pair, let $F_{\{n_0,i\}}$ be the set of non-increasing functions such that $(\forall n \geq n_o)[f(n) = i]$.

Let $g \in F_{\{n_0,i\}}$. We map g to the ordered $(i-1)$-tuple

$$\{\, g(1), g(2), \ldots, g(i-1) \,\}$$

Notice that this map is a bijection subject to the co-domain satisfying the conditions $g(j) \geq g(j+1)$ and $g(i-1) \geq i$.
The cardinality of the set $[g(1),\ g(2),\ \ldots,\ g(i-1)]$ is bounded by $\left|\mathbb{R}^{i-1}\right|$ and therefore by $|\mathbb{R}|$.
By the Cantor-Schroder-Bernstein theorem, $F_{\{n_0,i\}}$ is countable. Hence

$$S = \bigcup_{n_0} \bigcup_{i} F_{\{n_0,i\}}$$

is also countable.