CS21201 Discrete Structures
Practice Problems Solutions

Recurrence Relations

Q1. Solve the recurrence relation: $a_0 = 1$, $a_1 = 2$, and $a_n = \frac{2a_{n-1}^3}{a_{n-2}^2}$ for $n \geq 2$.

Solution 1: The solution provided utilizes a transformation $b_n = \log_2 a_n$ which yields the linear recurrence:

$$b_n = 3b_{n-1} - 2b_{n-2} + 1 \quad \text{with } b_0 = 0, b_1 = 1$$

- *Homogeneous Solution* $b_n^{(h)}$*:* The characteristic equation is $r^2 - 3r + 2 = 0$, with roots $r_1 = 1$ and $r_2 = 2$.
$$b_n^{(h)} = A_1(1)^n + A_2(2)^n = A_1 + A_2 2^n$$
- *Particular Solution* $b_n^{(p)}$*:*
$$b_n^{(p)} = nU1^n = Un$$
From the recurrence, we get $Un = 3u(n-1) - 2U(n-2) + 1 \Rightarrow U = -1$
- *General Solution:* $b_n = b_n^{(h)} + b_n^{(p)} = A_1 + A_2 2^n - n$. Applying initial conditions $b_0 = 0$ and $b_1 = 1$ gives $A_1 = -2$ and $A_2 = 2$.
$$b_n = -2 + 2 \cdot 2^n - n = 2^{n+1} - n - 2$$
The original sequence a_n is 2^{b_n}.
$$a_n = 2^{2^{n+1}-n-2} \quad \text{for } n \geq 0$$

Q2. For $n \geq 0$, let $S = \{1, 2, 3, \ldots, n\}$ (when $n = 0, S = \emptyset$), and let a_n denote the number of subsets of S that contain no consecutive integers. Find and solve a recurrence relation for a_n.

Solution 2: The initial conditions are $a_0 = 1$ (only $\emptyset$), $a_1 = 2$ $(\emptyset, \{1\})$, $a_2 = 3$ $(\emptyset, \{1\}, \{2\})$, $a_3 = 5$ $(\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\})$.

For $n \geq 2$, a valid subset $A \subseteq S = \{1, \ldots, n\}$ either:

- (a) $n \in A$**:** When this happens $n - 1 \notin A$ and $A \setminus \{n\}$ would be counted in a_{n-2} subsets.
- (b) $n \notin A$**:** For this case A would be counted in a_{n-1} .

Recurrence Relation: $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$, with initial conditions $a_0 = 1$ and $a_1 = 2$.

This is the Fibonacci sequence shifted, where $a_n = F_{n+2}$ (using the convention $F_1 = 1, F_2 = 1, F_3 = 2, \ldots$).

$$a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+2} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+2} \right] \quad \text{for } n \geq 0$$

Q3. Let $\Sigma = \{0, 1\}$ and $A = \{0, 01, 011, 1111\} \subseteq \Sigma^*$. For $n \geq 1$, let a_n count the number of strings in A^* of length n. Find and solve a recurrence relation of a_n.

Solution 3: The initial values are $a_1 = 1$ (0), $a_2 = 2$ $(00, 01)$, $a_3 = 4$ $(000, 001, 010, 011)$, and $a_4 = 9$.

For $n \geq 5$, a string of length n must end in a string from A.

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2a_{n-4} \quad \text{for } n \geq 5$$

- *Characteristic Equation:* $r^4 - r^3 - r^2 - r - 2 = 0$. This factors as $(r-2)(r+1)(r^2+1) = 0$.
- *Characteristic Roots:* 2, -1, and $\pm i$ (which give $\cos(n\pi/2)$ and $\sin(n\pi/2)$ terms).
- *General Solution:* $a_n = c_1(2)^n + c_2(-1)^n + c_3 \cos\left(\frac{n\pi}{2}\right) + c_4 \sin\left(\frac{n\pi}{2}\right)$.
- *Final Solution (after solving the system for c_1, c_2, c_3, c_4):*

$$a_n = \frac{8}{15}(2^n) + \frac{1}{6}(-1)^n + \frac{3}{10}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{10}\sin\left(\frac{n\pi}{2}\right) \quad \text{for } n \geq 1$$

Q4. Solve the recurrence relation $a_n - 3a_{n-1} = 5(7^n)$, where $n \geq 1$ and $a_0 = 2$.

Solution 4:

- *Homogeneous Solution $a_n^{(h)}$:* The characteristic equation is $r - 3 = 0$, so $r = 3$.

$$a_n^{(h)} = c(3^n)$$

- *Particular Solution $a_n^{(p)}$:* Since the forcing function is $f(n) = 5(7^n)$ and 7 is not a characteristic root, we seek a particular solution $a_n^{(p)} = A(7^n)$. Substituting this into the recurrence:

$$A(7^n) - 3A(7^{n-1}) = 5(7^n)$$

  Dividing by 7^{n-1} gives: $7A - 3A = 5(7)$, so $4A = 35$, which means $A = \frac{35}{4}$.

$$a_n^{(p)} = \frac{35}{4}7^n = \frac{5}{4}7^{n+1}$$

- *General Solution:* $a_n = c(3^n) + \frac{5}{4}7^{n+1}$.
- *Applying Initial Condition ($a_0 = 2$):* $2 = c(3^0) + \frac{5}{4}7^1 \Rightarrow 2 = c + \frac{35}{4} \Rightarrow c = 2 - \frac{35}{4} = \frac{8-35}{4} = -\frac{27}{4}$.
- *Final Solution:*

$$a_n = \frac{5}{4}7^{n+1} - \frac{27}{4}3^n = \frac{5}{4}7^{n+1} - \frac{1}{4}3^{n+3} \quad \text{for } n \geq 0$$

Q5. Pauline takes out a loan of S dollars that is to be paid back in T periods of time. If r is the interest rate per period for the loan, what (constant) payment Pauline must make at the end of each period?

Solution 5: Let a_n be the amount still owed on the loan at the end of the n-th period (following the n-th payment). The amount owed after the $(n+1)$-th period is the previous balance plus accrued interest minus the payment P(the payment she made at the end of the $(n+1)$st period) .

Recurrence Relation:

$$a_{n+1} = a_n + ra_n - P = (1+r)a_n - P \quad 0 \le n \le T-1$$

Initial Conditions: $a_0 = S$ (initial loan amount) and $a_T = 0$ (loan is fully repaid).

- *Homogeneous Solution* $a_n^{(h)}$*:* $a_n^{(h)} = c(1+r)^n$.
- *Particular Solution:* $a_n^{(p)} = A$ since no constant is a solution of the associated homogeneous relation. $A = (1+r)A - P \Rightarrow A - (1+r)A = -P \Rightarrow -rA = -P \Rightarrow A = P/r$.

$$a_n^{(p)} = \frac{P}{r}$$

- *General Solution:* $a_n = c(1+r)^n + \frac{P}{r}$. Using $a_0 = S$: $S = c(1+r)^0 + P/r \Rightarrow c = S - P/r$.

$$a_n = \left(S - \frac{P}{r}\right)(1+r)^n + \frac{P}{r}$$

- *Solving for* P *(using* $a_T = 0$*):*

$$0 = \left(S - \frac{P}{r}\right)(1+r)^T + \frac{P}{r}$$

$$-\frac{P}{r} = \left(S - \frac{P}{r}\right)(1+r)^T$$

$$\frac{P}{r}\left[(1+r)^T - 1\right] = S(1+r)^T$$

Final Payment Formula:

$$P = (Sr)\frac{(1+r)^T}{(1+r)^T - 1} = (Sr)\left[1 - (1+r)^{-T}\right]^{-1}$$

Q6. Determine the number of n-digit quaternary ($\{0,1,2,3\}$) sequences in which there is never a 3 anywhere to the right of a 0.

Solution 6: Let a_n be the count of such sequences of length n.

A valid sequence of length $n+1$ either:

- (i) Ends in $0, 1$, or 2 (3 possibilities): The first n symbols must form a valid sequence, counted by $3a_n$.
- (ii) Ends in 3: The first n symbols must not contain any 0's. These are sequences of length n over $\{1,2,3\}$, and there are 3^n such sequences.

Recurrence Relation: $a_{n+1} = 3a_n + 3^n$, with $a_0 = 1$ (for ϵ).

- *Homogeneous Solution* $a_n^{(h)}$*:* $a_n^{(h)} = A3^n$.
- *Particular Solution* $a_n^{(p)}$*:* Since 3 is a root, we seek $a_n^{(p)} = Bn3^n$. Substituting into $a_{n+1} = 3a_n + 3^n$:

$$B(n+1)3^{n+1} = 3(Bn3^n) + 3^n$$

Dividing by 3^n gives $3B(n+1) = 3Bn + 1 \Rightarrow 3Bn + 3B = 3Bn + 1 \Rightarrow 3B = 1 \Rightarrow B = \frac{1}{3}$.

$$a_n^{(p)} = \frac{1}{3}n3^n = n3^{n-1}$$

- *General Solution:* $a_n = A3^n + n3^{n-1}$. Using $a_0 = 1$: $1 = A3^0 + 0 \Rightarrow A = 1$.
- *Final Solution:*

$$a_n = 3^n + n3^{n-1} \quad \text{for } n \geq 0$$

Q7. Let $\Sigma = \{0, 1, 2, 3\}$. For $n \geq 1$, let a_n count the number of strings in Σ^n containing an odd number of 1's. Find and solve a recurrence relation for a_n.

Solution 7: For $n \geq 2$, consider the n-th symbol:

- (i) *n-th symbol is $0, 2$, or 3 (3 ways):* The prefix of length $n-1$ must have an odd number of 1's, counted by $3a_{n-1}$.
- (ii) *n-th symbol is 1 (1 way):* The prefix of length $n-1$ must have an even number of 1's. The total number of strings of length $n-1$ is 4^{n-1}. The number of strings with an even number of 1's is $4^{n-1} - a_{n-1}$.

Recurrence Relation: $a_n = 3a_{n-1} + (4^{n-1} - a_{n-1}) = 2a_{n-1} + 4^{n-1}$ for $n \geq 2$. *Initial Condition:* $a_1 = 1$ (only the string 1).

- *Homogeneous Solution $a_n^{(h)}$:* $r - 2 = 0 \Rightarrow r = 2$. $a_n^{(h)} = c(2^n)$.
- *Particular Solution $a_n^{(p)}$:* We seek $a_n^{(p)} = A4^{n-1}$. Substituting into the recurrence:

$$A4^{n-1} = 2(A4^{n-2}) + 4^{n-1}$$

  Multiplying by 4 and dividing by 4^{n-1}: $4A = 2A + 4 \Rightarrow 2A = 4 \Rightarrow A = 2$.

$$a_n^{(p)} = 2 \cdot 4^{n-1}$$

- *General Solution:* $a_n = c(2^n) + 2(4^{n-1})$. Using $a_1 = 1$: $1 = c(2^1) + 2(4^0) \Rightarrow 1 = 2c + 2 \Rightarrow 2c = -1 \Rightarrow c = -\frac{1}{2}$.
- *Final Solution:*

$$a_n = -\frac{1}{2}(2^n) + 2(4^{n-1}) = -\frac{1}{2}2^n + \frac{2}{4}4^n = \frac{1}{2}4^n - \frac{1}{2}2^n \quad \text{for } n \geq 1$$

Q8. Find an explicit formula for the sequences defined by this recurrence relation: $X_n = 2X_{n-1} + 15X_{n-2} + 2^n$, $X_1 = 2, X_2 = 4$.

Solution 8:

- *Homogeneous Solution $X_n^{(h)}$:* Characteristic equation $r^2 - 2r - 15 = 0$, which factors as $(r-5)(r+3) = 0$. Roots are $r_1 = 5$ and $r_2 = -3$.

$$X_n^{(h)} = C_1(5^n) + C_2(-3)^n$$

- *Particular Solution $X_n^{(p)}$:* Since 2 is not a characteristic root, we seek $X_n^{(p)} = A2^n$. Substituting into the recurrence:

$$A2^n = 2(A2^{n-1}) + 15(A2^{n-2}) + 2^n$$

  Dividing by 2^{n-2} gives $4A = 4A + 15A + 4 \Rightarrow 15A = -4 \Rightarrow A = -\frac{4}{15}$.

$$X_n^{(p)} = -\frac{4}{15}2^n$$

- *General Solution:* $X_n = C_15^n + C_2(-3)^n - \frac{4}{15}2^n$.

- Applying Initial Conditions:

$$X_1 = 2 \implies 2 = 5C_1 - 3C_2 - \frac{8}{15}$$
$$X_2 = 4 \implies 4 = 25C_1 + 9C_2 - \frac{16}{15}$$

Solving this system yields $C_1 = \frac{19}{60}$ and $C_2 = -\frac{19}{60}$.

- *Final Solution:*

$$X_n = \frac{19}{60}5^n - \frac{19}{60}(-3)^n - \frac{4}{15}2^n = -\frac{4 \cdot 2^n}{15} - \frac{(-1)^n 19 \cdot 3^n}{60} + \frac{19 \cdot 5^n}{60}$$

Q9. Find and solve a recurrence relation for the number of ways to park motorcycles and compact cars in a row of n spaces if each cycle requires one space and each compact needs two. (All motorcycles are identical in appearance, as are the cars, and we want to use up all the n spaces)

Solution 9: Let a_n be the number of ways to tile n spaces. Consider the last space:

- (i) If the last space is occupied by a motorcycle (1 space), the preceding $n-1$ spaces must be fully filled in a_{n-1} ways.
- (ii) If the last space is occupied by a compact car (2 spaces), the preceding $n-2$ spaces must be fully filled in a_{n-2} ways.

Recurrence Relation: $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. *Initial Conditions:* $a_0 = 1$ (empty lot, one way to be filled by zero vehicles) and $a_1 = 1$ (one motorcycle).

- This is the standard Fibonacci sequence with $a_n = F_{n+1}$. The characteristic equation is $r^2 - r - 1 = 0$, with roots $r_{1,2} = \frac{1 \pm \sqrt{5}}{2}$.
- *Final Solution :*

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Q10. Let $a_n, n \geq 0$ be the count of strings over $\{0, 1, 2\}$ containing no consecutive 1's and no consecutive 2's. Find a recurrence relation for a_n and solve it.

Solution 10: Let b_n, c_n, d_n denote the counts of the strings of length n that start with $0, 1, 2$, respectively. Let us also take $b_0 = 1$, $c_0 = 0$ and $d_0 = 0$ Let $a_n = b_n + c_n + d_n$ for all $n \geq 0$.

- $b_n = a_{n-1}$ for all $n \geq 1$ (A string starting with 0 can be followed by any valid string of length $n-1$).
- $c_n = b_{n-1} + d_{n-1} = a_{n-1} - c_{n-1}$ for all $n \geq 1$ (Starts with 1, must be followed by 0 or 2).
- $d_n = b_{n-1} + c_{n-1} = a_{n-1} - d_{n-1}$ for all $n \geq 1$ (Starts with 2, must be followed by 0 or 1).

Adding the three equations: $a_n = b_n + c_n + d_n$:

$$a_n = 3a_{n-1} - (c_{n-1} + d_{n-1}) = 3a_{n-1} - (a_{n-1} - b_{n-1}) = 2a_{n-1} + b_{n-1} = 2a_{n-1} + a_{n-2}$$

for all $n \geq 2$

Initial Conditions: $a_0 = 1$ (ϵ) and $a_1 = 3$ $(0, 1, 2)$.

- Characteristic equation $r^2 - 2r - 1 = 0$, with roots $r_{1,2} = 1 \pm \sqrt{2}$.

$$a_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n$$

- *Final Solution (after applying $a_0 = 1, a_1 = 3$):* $A = \frac{1+\sqrt{2}}{2}$ and $B = \frac{1-\sqrt{2}}{2}$.

$$a_n = \frac{1}{2}\left[(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right] \quad \text{for } n \geq 0$$

Q11. You start with a chocolate bar of size 1×10^{100}. In each operation, choose a break point uniformly at random among the integer positions, split the bar, keep the left piece and discard the right. Continue until a 1×1 piece remains. What is the probability that at some point the left piece has size 1×141421356237? (More generally, for a target size $k \geq 1$ what is the probability of ever seeing size k?)

Solution 11: Let $P(L)$ be the probability that, starting with a bar of length L, the process ever results in a left piece of size k.

- *Base Cases:* $P(k) = 1$ and $P(L) = 0$ for $L < k$.
- *Recurrence ($L > k$):* The first break point is uniform in $\{1, \ldots, L-1\}$,and only first pieces of length $\geq k$ can still reach k.

$$P(L) = \frac{1}{L-1} \sum_{i=k}^{L-1} P(i)$$

Let $S(L) = \sum_{i=k}^{L} P(i)$ be the cumulative sum. The recurrence can be rewritten as:

$$P(L) = \frac{S(L-1)}{L-1} \quad (L > k)$$

Then $S(L) = S(L-1) + P(L) = S(L-1) + \frac{S(L-1)}{L-1} = S(L-1)\left(1 + \frac{1}{L-1}\right) = \frac{L}{L-1}S(L-1)$. This relation telescopes to $S(L) = \frac{L}{L-1} \cdot \frac{L-1}{L-2} \cdots \frac{k+1}{k} S(k) = \frac{L}{k} S(k)$. Since $S(k) = P(k) = 1$, we have $S(L) = L/k$. The probability $P(L)$ is found by $P(L) = S(L) - S(L-1)$:

$$P(L) = \frac{L}{k} - \frac{L-1}{k} = \boxed{\frac{1}{k}} \quad \text{for } L \geq k$$

The probability of ever seeing the target size $k = 141421356237$ is:

$$\frac{1}{141421356237}$$

Recurrence Relations

Q1. Find a recurrence relation for the number of binary sequences of length n that have no consecutive 0s.

Solution: For $n \geq 1$ let a_n be the number of such sequences of length n. Let $a_n^{(0)}$ count those that end in 0, and $a_n^{(1)}$ those that end in 1. Then $a_n = a_n^{(0)} + a_n^{(1)}$.

If a sequence x of length $n-1$ ends in 1 (counted by $a_{n-1}^{(1)}$), we can append 0 or 1, contributing $2 \cdot a_{n-1}^{(1)}$ to a_n. If x ends in 0 (counted by $a_{n-1}^{(0)}$), we can only append 1, contributing $1 \cdot a_{n-1}^{(0)}$ to a_n. Thus, we have: $a_n = 2a_{n-1}^{(1)} + a_{n-1}^{(0)}$.

Since $a_{n-2} = a_{n-1}^{(1)}$ (a sequence of length $n-2$ is followed by 1 to create a sequence counted by $a_{n-1}^{(1)}$), we can substitute and simplify:

$$a_n = a_{n-1}^{(1)} + [a_{n-1}^{(1)} + a_{n-1}^{(0)}] = a_{n-1}^{(1)} + a_{n-1} = a_{n-1} + a_{n-2}$$

The recurrence relation is $a_n = a_{n-1} + a_{n-2}$, for $n \geq 3$, with initial conditions $a_1 = 2$ and $a_2 = 3$. (The sequences are $0, 1$ for $n = 1$ and $01, 10, 11$ for $n = 2$).

Q2. A string of decimal digits is considered to be a valid codeword if it contains an even number of 0 digits. For example, 02310023089 and 7254193776 are valid codewords, but 060796007620 is not valid. Let c_n denote the number of valid n-digit codewords.

(a) Derive, with clear justifications, a recurrence relation for c_n. Also supply the required number of initial conditions.

(b) Solve the recurrence relation of Part (a) to obtain a closed-form expression for c_n.

Solution: The initial condition is $c_0 = 1$ (the empty string contains zero, an even number of 0 digits).

For $n \geq 1$, a valid codeword W of length n can be obtained in two mutually exclusive ways:

(a) W starts with 0. The remaining $n-1$ digits must form an invalid codeword (containing an odd number of 0s). The total number of $n-1$ digit strings is 10^{n-1}, and c_{n-1} of them are valid. Thus, the count is $10^{n-1} - c_{n-1}$.

(b) W starts with a digit other than 0 (9 possibilities). The remaining $n-1$ digits must form a valid codeword. The count is therefore $9c_{n-1}$.

Summing these counts gives the recurrence relation:

$$c_n = (10^{n-1} - c_{n-1}) + 9c_{n-1} = 8c_{n-1} + 10^{n-1}$$

Q3. Solve the recurrence relation: $a_n = na_{n-1} + n(n-1)a_{n-2} + n!$ for $n \geq 2$, with $a_0 = 0$, $a_1 = 1$.

Solution: Dividing both sides of the given recurrence by $n!$, we get:

$$\frac{a_n}{n!} = \frac{na_{n-1}}{n!} + \frac{n(n-1)a_{n-2}}{n!} + \frac{n!}{n!}$$

$$\frac{a_n}{n!} = \frac{a_{n-1}}{(n-1)!} + \frac{a_{n-2}}{(n-2)!} + 1$$

Let $b_n = \frac{a_n}{n!}$. We have the new recurrence relation:

$$b_n = b_{n-1} + b_{n-2} + 1$$

The initial conditions become $b_0 = \frac{a_0}{0!} = \frac{0}{1} = 0$, and $b_1 = \frac{a_1}{1!} = \frac{1}{1} = 1$.

Homogeneous Solution: $b_n^{(h)} = A_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + A_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$ (similar to Fibonacci sequence).

Particular Solution: Assume $b_n^{(p)} = U \cdot 1^n = U$. From the recurrence, $U = U + U + 1 \Rightarrow U = -1$.

General Solution: $b_n = b_n^{(h)} + b_n^{(p)} = A_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + A_2 \left(\frac{1-\sqrt{5}}{2}\right)^n - 1$.

Using the initial conditions $b_0 = 0$ and $b_1 = 1$, we get $A_1 = \dfrac{3+\sqrt{5}}{2\sqrt{5}}$ and $A_2 = -\dfrac{3-\sqrt{5}}{2\sqrt{5}}$.

Therefore, $b_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n+2}\right] - 1$ for $n \geq 0$.

Finally, $a_n = n!\, b_n = \dfrac{n!}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{n+2} - \sqrt{5}\right]$.

Q4. An alphabet Σ consists of four numeric characters $1, 2, 3, 4$, and seven alphabetic characters a, b, c, d, e, f, g. Find and solve a recurrence relation for the number of words of length n (in Σ^*), where there are no consecutive (identical or distinct) alphabetic characters.

Solution: For $n \geq 0$ let a_n count the number of words of length n in Σ where there are no consecutive alphabetic characters. Let $a_n^{(1)}$ count those words that end with a numeric character, while $a_n^{(2)}$ counts those that end with an alphabetic character. Then $a_n = a_n^{(1)} + a_n^{(2)}$.

For $n \geq 1$: $a_{n+1} = a_{n+1}^{(1)} + a_{n+1}^{(2)}$.

$$a_{n+1}^{(1)} = 4a_n^{(1)} + 4a_n^{(2)} = 4a_n$$

$$a_{n+1}^{(2)} = 7a_n^{(1)}$$

$a_{n+1} = 4a_n + 7a_n^{(1)} = 4a_n + 7(4a_{n-1}) = 4a_n + 28a_{n-1}$

The initial conditions are $a_0 = 1$ and $a_1 = 11$. Now let $a_n = cr^n$ where $c, r \neq 0$ and $n \geq 0$. Then the resulting characteristic equation is $r^2 - 4r - 28 = 0$, where $r = \frac{4 \pm \sqrt{128}}{2} = 2 \pm 4\sqrt{2}$.

Hence $a_n = A[2+4\sqrt{2}]^n + B[2-4\sqrt{2}]^n$, $n \geq 0$.

$1 = a_0 \implies 1 = A + B$, and $11 = a_1 \implies 11 = A[2+4\sqrt{2}] + B[2-4\sqrt{2}]$.

$$11 = A[2+4\sqrt{2}] + (1-A)[2-4\sqrt{2}] = [2-4\sqrt{2}] + A[2+4\sqrt{2}-2+4\sqrt{2}] = [2-4\sqrt{2}] + 8\sqrt{2}A$$

So $A = \frac{9+4\sqrt{2}}{8\sqrt{2}}$, which simplifies to $A = \frac{8+9\sqrt{2}}{16}$ (based on your next line), and $B = 1 - A = \frac{8-9\sqrt{2}}{16}$.

Consequently,

$$a_n = \left[\frac{8+9\sqrt{2}}{16}\right][2+4\sqrt{2}]^n + \left[\frac{8-9\sqrt{2}}{16}\right][2-4\sqrt{2}]^n, \quad n \geq 0$$

Q5. Let a_n satisfy $a_1 = 1$ and for $n \geq 2$ the piecewise relation: $a_n = 2a_{n-1}$ if n is odd, and $a_n = 2a_{n-1} + 1$ if n is even. Develop a single recurrence relation for a_n that holds for both odd and even n, and solve it.

Solution: For both cases, $a_n - a_{n-2} = 2(a_{n-1} - a_{n-3})$

$$\Rightarrow a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$$

The characteristic equation is $r^3 - 2r^2 - r + 2 = 0$, with roots $r_1 = 2$, $r_2 = 1$, and $r_3 = -1$.

The general solution is $a_n = \gamma(2)^n + \beta(1)^n + \alpha(-1)^n$.

Using the initial conditions $a_1 = 1$, $a_2 = 2a_1 + 1 = 3$, and $a_3 = 2a_2 = 6$, and solving for the constants gives:

$$\alpha = \frac{1}{6}, \quad \beta = -\frac{1}{2}, \quad \gamma = \frac{5}{6}$$

The closed-form expression for a_n is:

$$a_n = \frac{5}{6} \cdot 2^n - \frac{1}{2} \cdot 1^n + \frac{1}{6} \cdot (-1)^n$$

Which can be written as:

$$a_n = \frac{1}{6}\left[5 \cdot 2^n + (-1)^n - 3\right]$$

Q6. Consider a 2×10 grid in which each cell initially contains a frog. Every frog simultaneously jumps to one of its edge-adjacent cells, chosen arbitrarily. In how many possible ways can the jumps occur so that after the jump, each cell again contains exactly one frog? More generally, answer the question for a $2 \times n$ grid.

Solution: Color the $2 \times n$ board in alternating black and white. Any jump goes from black to white or from white to black, so the black-squared frogs and the white-squared frogs can be handled independently. Let f_n be the number of valid patterns for one color.

- *Recurrence:* In the first column, the black frog can jump down (leaving $n-1$ columns) or right (forcing the next black frog to jump left, leaving $n-2$ columns).

$$f_n = f_{n-1} + f_{n-2} \quad (n \geq 3)$$

- *Initial Conditions:* $f_1 = 1$ and $f_2 = 2$.
- *Result:* Since the two colors are independent, the total number of valid simultaneous jump outcomes on a $2 \times n$ board is f_n^2.
- *For $n = 10$:* $f_{10} = 89$.
- *Total Ways:* $f_{10}^2 = 89^2 = 7921$.

Q7. Let $S = \{1, 2, 3, \ldots, 20\}$. Determine the number of ways to partition S into 10 unordered pairs such that, in each pair, the absolute difference of the two numbers is either 1 or 10.

Solution: Place the numbers $S = \{1, 2, \ldots, 20\}$ in a $2 \times n$ array with $n = 10$:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n+1 & n+2 & n+3 & \cdots & 2n \end{pmatrix}$$

An allowed pair is either a horizontal neighbor (difference 1) or a vertical neighbor (difference $n = 10$). Thus, a valid pairing of all $2n$ numbers corresponds exactly to a tiling of the $2 \times n$ board by 2×1 dominoes.

- *Recurrence:* Let $T(n)$ be the number of domino tilings of a $2 \times n$ board.

$$T(n) = T(n-1) + T(n-2) \quad (n \geq 3)$$

- *Initial Conditions:* $T(1) = 1$ and $T(2) = 2$.
- *Result:* $T(n)$ are the Fibonacci numbers F_n (where $F_1 = 1, F_2 = 2$).
- *Final Answer:* The number of required pairings for $n = 10$ is $T(10) = F_{10} = 89$.