CS21201: Discrete Structures

Autumn 2025

Practice Tutorial 2: Predicate Logic

Problem 1

(a) Given the predicate Person(x), which states that x is a person, and Ruler(x), which states that x is a ruler, write a statement in first-order logic that says “at most one person is a ruler.”
(b) Given the predicate Instant(i), which states that i is an instant in time, and Precedes(x, y), which states that x precedes y, write a sentence in first-order logic that says “time has a beginning, but has no end.”

Problem 2

Let p(x) be an open statement. Encode the following as quantified expressions.
(a) p(x) is true for exactly one value of x.
(b) p(x) is true for exactly two values of x.
(c) p(x) is true for at most one value of x.
(d) p(x) is true for at least two values of x.
(e) If p(x) is true for atleast two values of x, then p(x) is true for all values of x.
(f) If p(x) is true for atleast two values of x, then p(x) is true for all values of x.

Question 3

Consider the following statements:
C1: All lions are fierce creatures
C2: Some lions do not drink coffee
Indicate the truth of the following statements with appropriate proof/counterexample. You should use Predicate Logic with the predicates Lion(x), Fierce(x) and DrinksC(x).
a. It follows from C1 that there is a fierce creature
b. It follows from C1 and C2 that there is a fierce creature
c. It follows from C1 and C2, that some fierce creatures do not drink coffee.

Question 4

Solve the following problems using Predicate Logic by using predicates of your choice:

(a) No one respects a person who doesn't respect himself. No one will hire a person he does not respect. Therefore, a person who respects no one will never be hired by anyone.
(b) Tony, Mike and John belong to the Alpine Club. Every member of the Alpine Club is either a skier, or a mountain climber or both. No mountain climber likes rain, and all skiers like snow. Mike dislikes whatever Tony likes and likes whatever Tony dislikes. Tony likes rain and snow. Is there a member of the Alpine Club who is a mountain climber but not a skier? If so, who?

Practice Tutorial 2: Predicate Logic
(Solutions)

Question 1

Answer

a) At most one person is a ruler.
If two persons are ruler, then they must be the same person.
In first-order logic:

$\forall x \forall y [(person(x) \land Ruler(x) \land person(y) \land Ruler(y)) \longrightarrow x = y]$

b) Time has a beginning, but has no end.

for "Time has a beginning".
there exists a first instant, say b, such that nothing preceds it:

~~$\exists b [Instant(b) \land \forall i (Instant(i) \longrightarrow \sim preceder$~~

$\exists b [Instant(b) \land \forall i (Instant(i) \longrightarrow \sim Precedes(i,b))]$

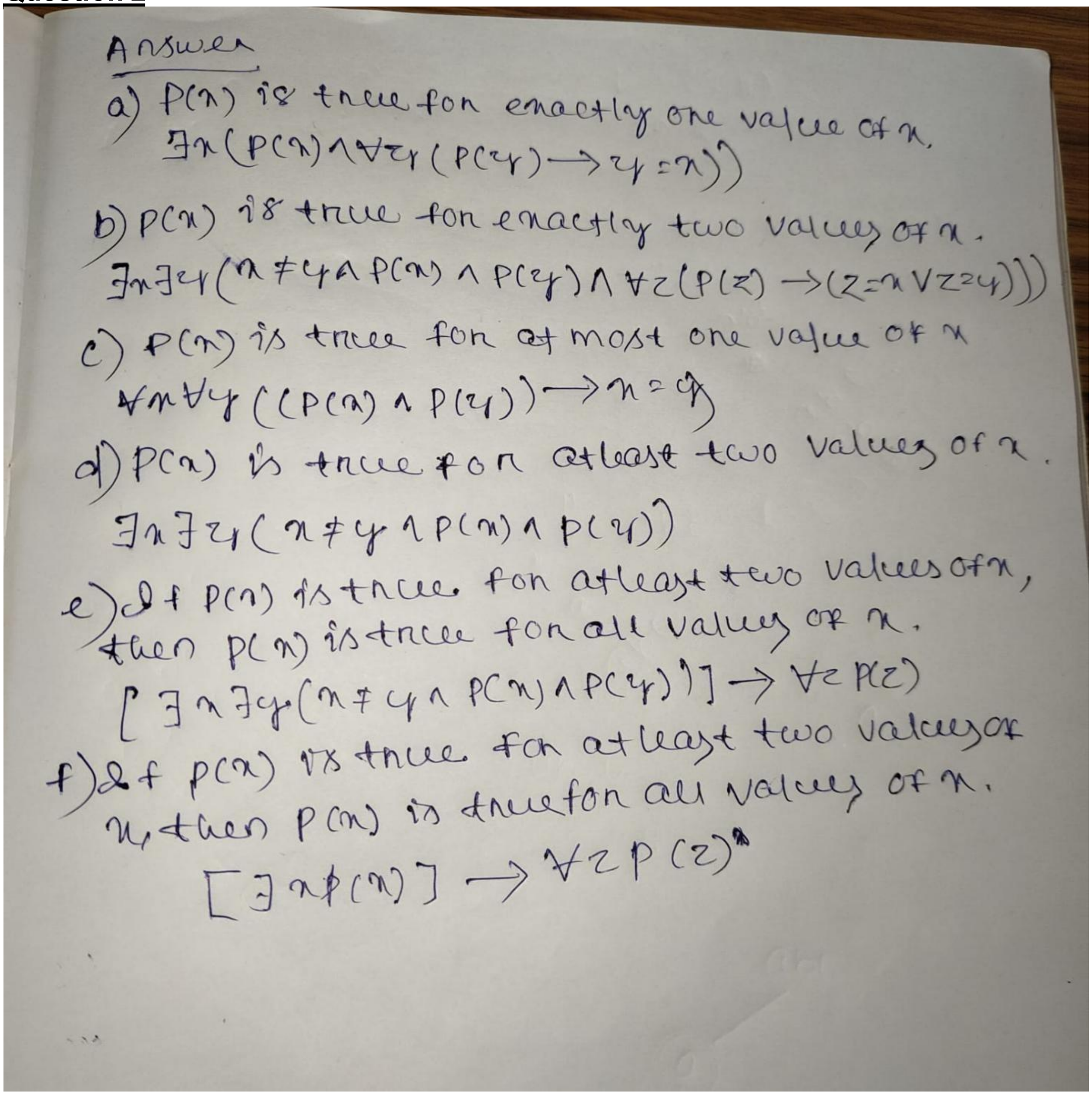
for "Time has no end".
for every instant i, there exist another instant j that comes after it.

$\forall i (Instant(i) \longrightarrow \exists j (Instant(j) \land precedes(i,j)))$

Final result will be

$[\exists b (Instant(b) \land \forall i (Instant(i) \longrightarrow \sim precedes(i,b)))]$
$\land [\forall i (Instant(i) \longrightarrow \exists j (Instant(j) \land precedes(i,j)))]$

Question 2



Answer

a) P(x) is true for exactly one value of x.

$\exists x(P(x) \wedge \forall y(P(y) \rightarrow y = x))$

b) P(x) is true for exactly two values of x.

$\exists x \exists y(x \neq y \wedge P(x) \wedge P(y) \wedge \forall z(P(z) \rightarrow (z = x \vee z = y)))$

c) P(x) is true for at most one value of x

$\forall x \forall y((P(x) \wedge P(y)) \rightarrow x = y)$

d) P(x) is true for atleast two values of x.

$\exists x \exists y(x \neq y \wedge P(x) \wedge P(y))$

e) If P(x) is true for atleast two values of x, then P(x) is true for all values of x.

$[\exists x \exists y(x \neq y \wedge P(x) \wedge P(y))] \rightarrow \forall z P(z)$

f) If P(x) is true for atleast two values of x, then P(x) is true for all values of x.

$[\exists x P(x)] \rightarrow \forall z P(z)$

Question 3

1. Coding of the statements is as under (∈ Creatures)

   S1: $\forall$ *[in() → ir()]*

   S2: $\exists$ *[in() ∧ ¬rins()]*

   a. $\exists$ *ir()*

      This statement is true if there is at least one lion. We cannot say that this directly follows from S1. This statement is **False.**

b. From S2, we notice that ∃ *[in() ∧ ¬rins()]*. So let that creature be . We know that is a lion. From S1, any creature who is a lion is fierce. Therefore is fierce. By existential generalization, ∃ *ir()*. Hence this statement is **True.**

c. ∃ *[ir() ∧ ¬rins()]*: Notice that similar to (b), we derive that is a lion and does not drink coffee. From S1, all lions are fierce. Therefore is fierce. This implies that *ir()* ∧ ¬*rins()*, by existential generalization, ∃ *[ir() ∧ ¬rins()]*. The statement is **True.**

Question 4

(a) Predicates used are $Respect(x, y)$: person x respects person y and $Hire(x, y)$: person x hires person y.

S1 : $\forall x\,(\neg Respect(x, x) \rightarrow \neg\exists y(Respect(y, x))$

S2 : $\forall x \forall y\,(\neg Respect(x, y) \rightarrow \neg Hire(x, y)) \equiv \forall x \forall y\,(Hire(x, y) \rightarrow Respect(x, y))$

G : $\forall x\,[(\neg\exists y\, Respect(x, y)) \rightarrow (\neg\exists z\, Hire(z, x))]$

Simplification of G:

$\forall x\,[(\neg\exists y\, Respect(x, y)) \rightarrow (\neg\exists z\, Hire(z, x))]$	(1)
$\forall x\,[(\exists z\, Hire(z, x)) \rightarrow (\exists y\, Respect(x, y))]$	(2) Contrapositive (1)

Proof by contradiction, assume that $\neg G$ is true.

$\neg\forall x\,[(\exists z\, Hire(z, x)) \rightarrow (\exists y\, Respect(x, y))]$	(3)
$\exists x\, \neg[\neg(\exists z\, Hire(z, x)) \vee (\exists y\, Respect(x, y))]$	(4) Properties of ¬ and →
$\exists x\,[(\exists z\, Hire(z, x)) \wedge \neg(\exists y\, Respect(x, y))]$	(5) De Morgan's Laws
Instantiate (5), by $x = A$ and $z = B$	
$Hire(B, A)$	(6)
$\neg(\exists y\, Respect(A, y))$	(7)
Instantiate S2 by $x = B$ and $z = A$	(8)
$Hire(B, A) \rightarrow Respect(B, A)$	
$Respect(B, A)$	(9) Modus Ponens (7, 8)
$\forall x(\exists y\,(Respect(y, x)) \rightarrow Respect(x, x))$	(10) Contrapositive (S1)
Instantiate by $x = A$ and $y = B$	
$Respect(B, A) \rightarrow Respect(A, A)$	(11)
$Respect(A, A)$	(12) Modus Ponens(9, 11)

But from (7), $\neg(\exists y\, Respect(A, y)) \Rightarrow \forall y\, \neg Respect(A, y) \Rightarrow \neg Respect(A, A)$

Hence we have a contradiction

(b) Predicates:

$a(x)$: Person x belongs to the Alpine Club
$s(x)$: Person x is a skier
$m(x)$: Person x is a mountain climber
$l(x, y)$: Person x likes weather event y

Statements:

S1 : $a(Tony) \wedge a(Mike) \wedge A(John)$
S2 : $\forall x\, [a(x) \rightarrow (s(x) \vee m(x))]$
S3 : $\neg\exists x\, [m(x) \wedge l(x, Rain)]$
S4 : $\forall x[s(x) \rightarrow l(x, Snow)]$
S5 : $\forall y[l(Mike, y) \leftrightarrow \neg l(Tony, y)]$
S6 : $l(Tony, Rain) \wedge l(Tony, Snow)$

Since Tony likes both Rain and Snow and Mike dislikes whatever Tony likes and likes whatever Tony dislikes
Mike does not like Rain and Mike does not like Snow

$\neg l\,(Mike, Rain)$ (1)
$\neg l\,(Mike, Snow)$ (2)

From S4, instantiating $x = Mike$, we get
$s(Mike) \rightarrow l(Mike, Snow)$ (3)
$\neg s(Mike)$ (4) Modus Tollens(2, 3)

From S2, instantiating $x = Mike$, we get
$a(Mike) \rightarrow (s(Mike) \vee m(Mike))$ (5)
$s(Mike) \vee m(Mike)$ (6) Modus Ponens(S1, 5)
$m(Mike)$ (7) (4, 6)

Clearly **Mike** is a Mountain Climber and not a skier, from (4) and (7).