1.

a.

We have

$$
\begin{aligned}
A(x) &= a_0 + a_1 x + a_2 x^2 + \sum_{n\geqslant 3} a_n x^n \\
&= x + 2x^2 + \sum_{n\geqslant 3} (2a_{n-2} + a_{n-3} + 2)x^n \\
&= x + 2x^2 + 2x^2 \sum_{n\geqslant 3} a_{n-2}x^{n-2} + x^3 \sum_{n\geqslant 3} a_{n-3}x^{n-3} + 2x^3 \sum_{n\geqslant 3} x^{n-3} \\
&= x + 2x^2 + 2x^2\left(A(x) - 0\right) + x^3 A(x) + \frac{2x^3}{1-x} \\
&= (2x^2 + x^3)A(x) + \frac{x+x^2}{1-x}.
\end{aligned}
$$

$$
A(x) = \frac{x+x^2}{(1-x)(1-2x^2-x^3)} = \frac{x(1+x)}{(1-x)(1+x)(1-x-x^2)} = \frac{x}{(1-x)(1-x-x^2)}.
$$

b.

We can write

$$
\frac{x}{(1-x)(1-x-x^2)} = \frac{A}{1-x} + \frac{Bx+c}{1-x-x^2}.
$$

Solving gives $A = -1$ and $B = C = 1$, that is,

$$
A(x) = \frac{1+x}{1-x-x^2} - \frac{1}{1-x}.
$$

The OGF of the Fibonacci sequence is $\frac{x}{1-x-x^2}$, that is, $\frac{x}{1-x-x^2}$ generates $F_0, F_1, F_2, \ldots, F_n, \ldots$. This implies that $\frac{1}{1-x-x^2}$ generates $F_1, F_2, F_3, \ldots, F_{n+1}, \ldots$. Finally, $\frac{1}{1-x}$ generates $1, 1, 1, \ldots$. Therefore, we have $a_n = F_n + F_{n+1} - 1 = F_{n+2} - 1$ for all $n \geqslant 0$.

2.

Let us compute the first few terms of the sequence: $a_0 = 2$, $a_1 = 2$, $a_2 = 10$, $a_3 = 26$, $a_4 = 82$, $a_5 = 242$, $\ldots$. From this, a pattern emerges: $a_n = 3^n + (-1)^n$. This can be established by induction on n. This is true for $n = 0, 1$. For $n \geqslant 2$, we have $a_n = 2 \times (3^{n-1} + (-1)^{n-1}) + 3 \times (3^{n-2} + (-1)^{n-2}) = 3^n + (-1)^n$. Therefore $a_n = 3^n + (-1)^n \leqslant 3^n + 1$ for all $n \geqslant 0$. So the desired answer is $k = 3$.

3.

Consider each package of 25 envelopes as one unit. Then the answer to the problem is the coefficient of x^{120} in $(x^6+x^7+\ldots+x^{39}+x^{40})^4 = x^{24}(1+x+\ldots+x^{34})^4$. This is the same as the coefficient of x^{96} in $[(1-x^{35})/(1-x)]^4 = (1-x^{35})^4(1-x)^{-4} = [1-4x^{35}+6x^{70}-\ldots+x^{140}][\binom{-4}{0}+\ldots+\binom{-4}{26}(-x)^{26}+\ldots+\binom{-4}{61}(-x)^{61}+\ldots+\binom{-4}{96}(-x)^{96}+\ldots]$.

Consequently the answer is $\binom{-4}{96}(-1)^{96}-4\binom{-4}{61}(-1)^{61}+6\binom{-4}{26}(-1)^{26} = \binom{99}{96}-4\binom{64}{61}+6\binom{29}{26}$.

4.

$(1-4x)^{-1/2} = [\binom{-1/2}{0}+\binom{-1/2}{1}(-4x)+\binom{-1/2}{2}(-4x)^2+\ldots]$. The coefficient of x^n is $\binom{-1/2}{n}(-4)^n =$

$$\frac{((-1/2)-n+1)((-1/2)-n+2)\cdots((-1/2)-1)(-1/2)}{n!}(-4)^n =$$

$$\frac{(1+2n-2)(1+2n-4)\cdots(1+2)(1)}{n!}(2)^n =$$

$$\frac{(2n-1)(2n-3)\cdots(5)(3)(1)}{n!}(2)^n =$$

$$\frac{[(2n-1)(2n-3)\cdots(5)(3)(1)](2^n)(n!)}{n!n!} = \frac{(2n)!}{n!n!} = \binom{2n}{n}.$$

5.

(a) $f(x) = [1/(1-x^2)][1/(1-x^4)][1/(1-x)^6]\cdots = \prod_{i=1}^{\infty}[1/(1-x^{2i})]$

(b) $g(x) = (1+x^2)(1+x^4)(1+x^6)\cdots = \prod_{i=1}^{\infty}(1+x^{2i})$

(c) $h(x) = (1+x)(1+x^3)(1+x^5)\cdots = \prod_{i=1}^{\infty}(1+x^{2i-1})$

6.

(a) 0

(b) $\binom{-3}{12}(-1)^{12}-5\binom{-3}{14}(-1)^{14} = \binom{14}{12}-5\binom{16}{14}$

(c) $\binom{-4}{15}(-1)^{15}+\binom{4}{1}\binom{-4}{14}(-1)^{14}+\binom{4}{2}\binom{-4}{13}(-1)^{13}+\binom{4}{3}\binom{-4}{12}(-1)^{12}+\binom{4}{4}\binom{-4}{11}(-1)^{11} =$
$\binom{18}{15}+\binom{4}{1}\binom{17}{14}+\binom{4}{2}\binom{16}{13}+\binom{4}{3}\binom{15}{12}+\binom{14}{11}$

7.

(a) $(x^3+x^4+\ldots)^4 = x^{12}(1+x+x^2+\ldots)^4 = x^{12}(1-x)^{-4}$. The coefficient of x^{12} in $(1-x)^{-4}$ is $\binom{-4}{12}(-1)^{12} = (-1)^{12}\binom{4+12-1}{12}(-1)^{12} = \binom{15}{12}$.

(b) $(x^3+x^4+\ldots+x^9)^4 = x^{12}(1+x+x^2+\ldots+x^6)^4$. The coefficient of x^{12} in $[(1-x)^7/(1-x)]^4 = (1-x^7)^4(1-x)^{-4} = [1-4x^7+\ldots+x^{28}][\binom{-4}{0}+\ldots+\binom{-4}{5}(-x)^5+\ldots+\binom{-4}{12}(-x)^{12}+\ldots]$

is $(-4)\binom{-4}{5}(-1)^5 + \binom{-4}{12}(-1)^{12} = (4)(-1)^5\binom{8}{5} + \binom{15}{12} = \binom{15}{12} - 4\binom{8}{5}$.

8.

(a) The coefficient of x^{24} in $(x^2+x^3+\ldots)^5 = x^{10}(1+x+x^2+\ldots)^5 = x^{10}(1-x)^{-5} = x^{10}[\binom{-5}{0}+\binom{-5}{1}(-x)+\binom{-5}{2}(-x)^2+\ldots]$ is $\binom{-5}{14}(-1)^{14} = (-1)^{14}\binom{5+14-1}{14}(-1)^{14} = \binom{18}{14}$. This is the number of ways to distribute the 24 bottles of one type of soft drink among the surveyors so that each gets at least two bottles. Since there are two types, the two cases can be distributed according to the given restrictions in $\binom{18}{14}^2$ ways.

(b) The coefficient of x^{24} in $(x^3+x^4+\ldots)^5$ is $\binom{13}{9}$ and the answer is $\binom{18}{14}\binom{13}{9}$.

9.

$(x+x^2+x^3+x^4+x^5+x^6)^{12} = x^{12}[(1-x^6)/(1-x)]^{12} = x^{12}((1-x)^6)^{12}[\binom{-12}{0}+\binom{-12}{1}(-x)+\binom{-12}{2}(-x)^2+\ldots]$. The numerator of the answer is the coefficient of x^{18} in $(1-x^6)^{12}[\binom{12}{0}+\binom{-12}{1}(-x)+\ldots] = [1-\binom{12}{1}x^6+\binom{12}{2}x^{12}-\binom{12}{3}x^{18}+\ldots+x^{72}][\binom{-12}{0}+\binom{-12}{1}(-x)+\ldots]$ and this equals $\binom{-12}{18}(-1)^{18}-\binom{12}{1}\binom{-12}{12}(-1)^{12}+\binom{12}{2}\binom{-12}{6}(-1)^6-\binom{12}{3}\binom{-12}{0} = \binom{29}{18}-\binom{12}{1}\binom{23}{12}+\binom{12}{2}\binom{17}{6}-\binom{12}{3}$. The final answer is obtained by dividing the last result by 6^{12}, the size of the sample space.

10.

(a) There are $2^{8-1} = 2^7$ compositions of 8 and $2^{\lfloor 8/2 \rfloor} = 2^4$ palindromes of 8. Assuming each composition of 8 has the same probability of being generated, the probability a palindrome of 8 is generated is $2^4/2^7 = 1/8$.

(b) Assuming each composition of n has the same probability of being generated, the probability a palindrome of n is generated is $2^{\lfloor n/2 \rfloor}/2^{n-1} = 2^{\lfloor n/2 \rfloor - n+1} = 2^{1-\lceil n/2 \rceil}$.

11.

Let the discrete random variable Y count the number of tosses Leroy makes until he gets the first tail. Then $Pr(Y = y) = (\frac{2}{3})^{y-1}(\frac{1}{3})$, $y = 1, 2, 3, \ldots$.

Here we are interested in $Pr(Y = 1) + Pr(Y = 3) + Pr(Y = 5) + \cdots = (\frac{1}{3}) + (\frac{2}{3})^2(\frac{1}{3}) + (\frac{2}{3})^4(\frac{1}{3}) + \cdots = (\frac{1}{3})[1 + (\frac{2}{3})^2 + (\frac{2}{3})^4 + \cdots] = (\frac{1}{3})\frac{1}{1-(\frac{2}{3})^2} = (\frac{1}{3})\frac{1}{5/9} = (\frac{1}{3})(\frac{9}{5}) = \frac{3}{5}$.

12.

(a) $-27, 54, -36, 8, 0, 0, 0, \ldots$ (b) $0, 0, 0, 0, 1, 1, 1, 1, 1, \ldots$
(c) $f(x) = x^3/(1 - x^2) = x^3[1 + x^2 + x^4 + x^6 + \ldots] = x^3 + x^5 + x^7 + x^9 + \ldots$, so $f(x)$ generates the sequence $0, 0, 0, 1, 0, 1, 0, 1, 0, 1, \ldots$
(d) $f(x) = 1/(1+3x) = 1 + (-3x) + (-3x)^2 + (-3x)^3 + \ldots$, so $f(x)$ generates the sequence $1, -3, 3^2, -3^3, \ldots$

T1.

We have

$$\begin{aligned} A(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \\ &= 1 + \sum_{n \geqslant 1} (a_{n-1} + 2a_{n-2} + 3a_{n-3} + \cdots + na_0)x^n \\ &= 1 + x(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)(1 + 2x + 3x^2 + 4x^3 + \cdots) \\ &= 1 + \frac{xA(x)}{(1-x)^2}. \end{aligned}$$

Simplification gives

$$A(x) = \frac{(1-x)^2}{1 - 3x + x^2}.$$

a.

We have

$$A(x) = \frac{(1-x)^2}{1-3x+x^2} = 1 + \frac{x}{1-3x+x^2} = 1 + \frac{x}{(1-\alpha x)(1-\beta x)},$$

where $\alpha = \frac{3+\sqrt{5}}{2}$ and $\beta = \frac{3-\sqrt{5}}{2}$. Now, write

$$\frac{x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x},$$

that is,

$$x = A(1-\beta x) + B(1-\alpha x).$$

Equating the constant term from both sides gives $A + B = 0$, that is, $B = -A$. Then we equate the coefficient of x from both sides to get $1 = (\alpha - \beta)A$. This gives $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$. We therefore have

$$a_n = \begin{cases} 1 & \text{if } n = 0, \\ \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right] & \text{if } n \geqslant 1. \end{cases}$$

b.

c.

We have

$$F_{2n} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n}\right].$$

Finally, note that $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$, and $\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2}$.

T2.

(a) $f(x) = (1+x+x^2+\ldots+x^5)(1+x^2+x^4+\ldots+x^{10})\cdots = \prod_{i=1}^{\infty}(1+x^i+x^{2i}+\ldots+x^{5i}) = \prod_{i=1}^{\infty}[(1-x^{6i})/(1-x^i)]$

(b) $\prod_{i=1}^{12}(1+x^i+x^{2i}+\ldots+x^{5i}) = \prod_{i=1}^{12}[(1-x^{6i})/(1-x^i)]$

T3.

(a) If a palindrome of 11 starts with 1, then that palindrome ends in 1. Upon removing '1+' from the start and '+1' from the end of the palindrome, we find a palindrome of 9. And there are $2^{\lfloor 9/2 \rfloor} = 2^4 = 16$ palindromes of 9.

Similar arguments tells us that there are $2^{\lfloor 7/2 \rfloor} = 8$ palindromes of 11 that start with 2, $2^{\lfloor 5/2 \rfloor} = 4$ palindromes of 11 that start with 3, and $2^{\lfloor 3/2 \rfloor} = 2$ palindromes of 11 that start with 4.

(b) For the palindromes of 12, we find that $2^{\lfloor 10/2 \rfloor} = 32$ start with 1, $2^{\lfloor 8/2 \rfloor} = 16$ start with 2, $2^{\lfloor 6/2 \rfloor} = 8$ start with 3, and $2^{\lfloor 4/2 \rfloor} = 4$ start with 4.