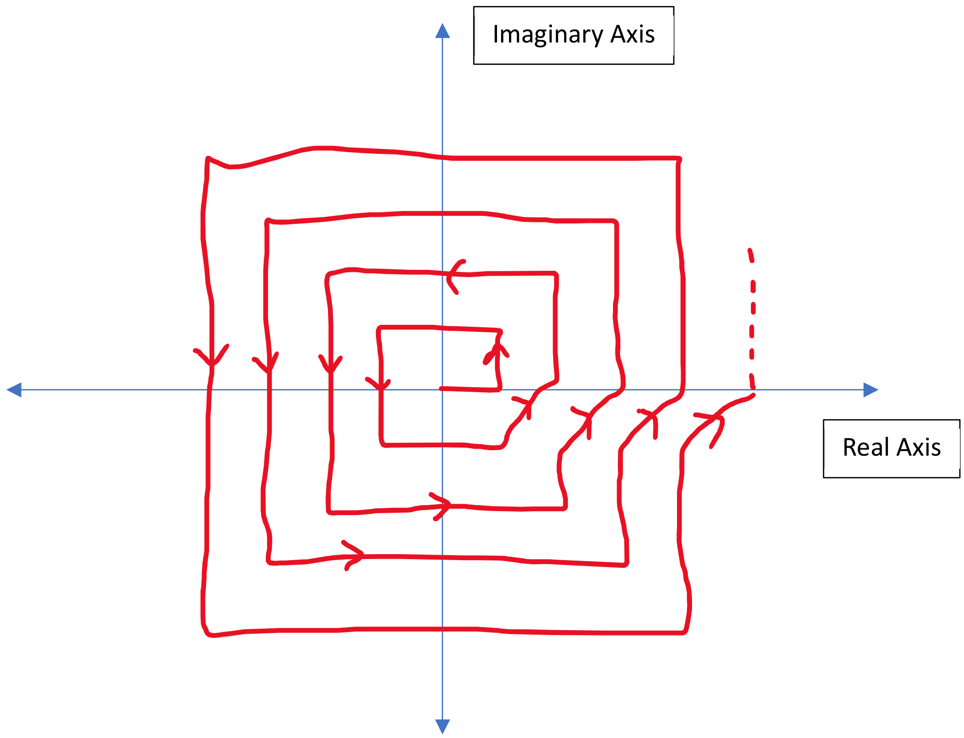
Set Sizes

1. Prove that the set $A = \{a + ib \mid a, b \in \mathbb{Z}\}$ of Gaussian Integers is countable.

   The bijection is given by the exhaustive enumeration shown below in terms of the coordinate system, note that the same element is not visited again.



An alternate way could be to see that $|A| = |\mathbb{Z} \times \mathbb{Z}|$. Then we could prove that $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{Z}|$. Without loss of generality, we need to prove that the cartesian product of two countable sets A and B is countable.

Proof: $\forall a \in A$, define $B_a = \{(a, b) \mid b \in B\}$. Clearly $f: B \to B_a$ is a bijection as we can map each instance $b \in B$ to $(a, b) \in B_a$. Hence each B_a is countable.
By definition:

$$A \times B = \bigcup_{a \in A} B_a$$

Since the union of countably many countable sets is countable, $A \times B$ is countable.

2. **[Sets of Functions]** We have the following sets, determine whether they are countable or uncountable.

   a) The set of all functions from $\mathbb{N}$ to $\{1, 2\}$
   Uncountable
   Assume that this set (S) is countable. Therefore, there is a bijective mapping $f : \mathbb{N} \to S$ such that $a \in \mathbb{N} \to f_a \in S$. Consider the following diagonalization argument:

	f_1	f_2	f_3	f_4	..	..	f_n	..	..
1	**1**	2	2	1	..	..	2	..	..
2	1	**1**	2	1	..	..	1	..	..
3	2	2	**1**	2	..	..	2	..	..
4	1	1	1	**2**	..	..	1	..	..
..	..	..	..	..	..	..	..	..	..
..									
n	2	1	2	2	..	..	**1**	..	..
..									

Consider a function F which gives outputs opposite of those given by f_i across the diagonal above.

$$F(x) = \begin{cases} 1 & , f_x(x) = 2 \\ 2 & , f_x(x) = 1 \end{cases} \quad \forall x \in \mathbb{N}$$

$F \in S$. However, this means that $F = f_k$ for some k. But, if $f_k(k) = 1$, then $F(k) = 2$. Hence our assumption is invalid.

b) The set of all functions from $\mathbb{N}$ to $\mathbb{N}$

Uncountable, any superset of an uncountable set is also uncountable. Alternatively, the above diagonalization argument can be modified to prove this statement by selecting $F(x) = f_x(x) + 1$.

c) The set of all functions from $\{1, 2\}$ to $\mathbb{N}$

Countable, notice that this set will have the same cardinality as $\mathbb{N} \times \mathbb{N}$. Let this set be S. Each $f \in S$ has two values $f(1)$ and $f(2)$, both of which are in $\mathbb{N}$.

d) The set of all non-increasing functions from $\mathbb{N}$ to $\mathbb{N}$

Countable, this set is the countable union of countable sets. Let this set be denoted by S. Let $f \in S$. Then:

$$f(1) \geq f(2) \geq f(3) \geq \cdots$$

Eventually, $\forall f \in S$, there exist values i and n_0 such that $\forall n \geq n_o[f(n) = i]$. Mathematically,

$$\exists n_0 \exists i \; (\forall n \geq n_o)[f(n) = i]$$

For a fixed (n_o, i) pair, let $F_{\{n_0,i\}}$ be the set of non-increasing functions such that $(\forall n \geq n_o)[f(n) = i]$.

Let $g \in F_{\{n_0,i\}}$. We map g to the ordered $(i-1)$-tuple

$$\{ g(1), g(2), \ldots, g(i-1) \}$$

Notice that this map is a bijection subject to the co-domain satisfying the conditions $g(j) \geq g(j+1)$ and $g(i-1) \geq i$.
The cardinality of the set $[g(1),\ g(2),\ \ldots,\ g(i-1)]$ is bounded by $|\mathbb{R}^{i-1}|$ and therefore by $|\mathbb{R}|$.
By the Cantor-Schroder-Bernstein theorem, $F_{\{n_0,i\}}$ is countable. Hence

$$S = \bigcup_{n_0} \bigcup_{i} F_{\{n_0,i\}}$$

is also countable.

3. Prove that the set of all permutations of $\mathbb{N}$ is not countable.

The set of all permutations of $\mathbb{N}$ is equinumerous with the set of all bijective functions $f: \mathbb{N} \to \mathbb{N}$. Notice that each permutation is basically a function mapping a place (which is a natural number) to a natural number.
It is easy to see using a diagonalization argument that the set of all bijective functions $f: \mathbb{N} \to \mathbb{N}$ is uncountable.

4. Let A be an infinite set.

   a) Prove that there is a map $A \to A$ which is injective but not surjective.

   Pick any countable subset $B = \{b_1, b_2, b_3, \dots, \}$ of A. Define the map $f: A \to A$ as
$$f(a) = \begin{cases} b_{n+1} & if\ a = b_n \\ a & otherwise \end{cases}$$

   b) Prove that there is a map $A \to A$ which is surjective but not injective.

   Let B be the same as in (a). Pick the map $g: A \to A$ as
$$f(a) = \begin{cases} b_1 & if\ a = b_1 \\ b_{n-1} & if\ a = b_n\ for\ n \geq 2 \\ a & otherwise \end{cases}$$

5. Consider the set $S = \{a + b\sqrt{7} \mid a, b \in \mathbb{Z}\}$. Prove that $\mathbb{R} - S$ is uncountable.

Note that S is countable, since $f : \mathbb{Z} \times \mathbb{Z} \to S$ is a bijection. Assume that $\mathbb{R} - S$ is countable, then $(\mathbb{R} - S) \cup S = \mathbb{R}$ is countable, since it will be the union of two countable sets. However, $\mathbb{R}$ is uncountable and therefore our assumption is wrong.

By the same logic, for any countable set $S, \mathbb{R} - S$ is uncountable.

6. Prove that the union of two sets equinumerous with $\mathbb{R}$ is again equinumerous with $\mathbb{R}$.

Consider A and B such that $|A| = |B| = |\mathbb{R}|$. It is obvious that $|A| = |\mathbb{R}| \leq |A \cup B|$. We therefore prove that $|A \cup B| \leq |\mathbb{R}|$. We provide an injective map $f : A \cup B \to [0, 2]$. It is already known that $|\mathbb{R}| = |[0, 1]| = |[0, 2]|$.
Let $g_1: A \to [0, 1]$ and $g_2: B \to [1, 2]$.
Let $A = \{a_1, a_2, a_3, a_4, \dots\}$ $B = \{b_1, b_2, b_3, b_4, \dots.\}$. and $A \cup B = \{a_1, b_1, a_2, a_3, b_3, \dots\}$

Define the injective map $f : A \cup B \to [0, 2]$ as:
$$f(x) = \begin{cases} g_1(x), & x \in A \\ g_2(x), & x \notin A \end{cases}$$

Is this map bijective?

7. Provide an explicit bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. It should not be an exhaustive enumeration.

Think about the exhaustive enumeration method where you tried to enumerate all the coordinates in a diagonal scheme.
For a natural number N, try to find out how many diagonals can be completely enumerated.

First diagonal enumerates (1, 1)	: 1 point
Second diagonal enumerates (2, 1) and (1, 2)	: 2 points

Third diagonal enumerates (3, 1), (2, 2) and (1, 3) : 3 points
nth diagonal enumerates (n, 1), ($n - 1$, 2) …. (1, n) : n points

This is the maximum n such that $\frac{n(n+1)}{2} \leq N$. Find this n. Let $d = N - n$
$[d = 0]$ The mapping is given by $N \rightarrow (1, n)$
$[0 < d \leq n + 1]$ The mapping is given by $N \rightarrow \big(n + 1 - (d - 1), 1 + (d - 1)\big) = (n - d + 2, d)$

8. Determine whether the following sets are countable or uncountable:
   a) The set of all finite subsets of $\mathbb{N}$
      Countable
      Let this set be A. We find a way to enumerate all the finite subsets of $\mathbb{N}$ using the following enumeration scheme

{}					
{1}	{2}	{3}	{4}	{5}	{6}
{1, 2}	{1, 3}	{1, 4}	{1, 5}	{1, 6}	{1, 7}
{1, 2, 3}	{1, 2, 4}	{1, 2, 5}	{1, 2, 6}	{1, 2, 7}	…
{1, 2, 3, 4}	{1, 2, 3, 5}	{1, 2, 3, 6}	{1, 2, 3, 7}	{1, 2, 3, 8}	…
{1, 2, 3, 4, 5}	{1, 2, 3, 4, 6}	{1, 2, 3, 4, 7}	{1, 2, 3, 4, 8}	{1, 2, 3, 4, 9}	…

      Alternate Method: We also know that a countable union of countable sets is countable. The above set can be written as $S = S_0 \cup S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_n \cup \ldots$ where $S_i \subseteq \mathbb{N}^i$. Each of $\mathbb{N}^i$ is countable.

   b) The set of all infinite subsets of $\mathbb{N}$
      Uncountable
      Let this set be B. Assume that B is countable, then $A \cup B$ is also countable, since it is a union of a countable number of countable sets. But $A \cup B$ is the number of subsets of $\mathbb{N}$ which is obviously uncountable. Therefore, our assumption is wrong and B is uncountable.

9. Infinite Bit Sequences] As the name suggests, an infinite bit sequence is an infinite sequence of 0s and 1s. Denote S as the set of all infinite bit sequences. Let $\alpha(n)$ be the nth element of an infinite bit sequence $\alpha \in S$. Determine whether the following sets are countable or uncountable:
   a) S
      Uncountable
      A simple diagonalization argument would suffice. Construct an infinite bite sequence β such that $\beta(n) = \overline{\alpha_n(n)}$, where α_n denotes the α which is mapped to integer $n \in \mathbb{N}$. Clearly, $\beta \in S$. Let $\beta = \alpha_k$ for some $k \in \mathbb{N}$. But $\beta(k) \neq \alpha_k(k)$ by virtue of the above construction. Hence S is uncountable

   b) $T_1 = \{\alpha \in S \mid \alpha(n) = 1$ and $\alpha(n + 1) = 0$ for some $n \geq 0\}$
      Uncountable
      Consider the set $T_3 = \{\alpha \in S \mid \alpha(0) = 1$ and $\alpha(1) = 0\}$. We have $T3 \subseteq T1$, and so $|T_3| \leq |T_1| \leq |S|$ (use the canonical inclusion maps which are injective).
      On the other hand, take any $\alpha = (1, 0, a_2, a_3, \ldots, a_n, \ldots,) \in T_3$. The map taking $\alpha \rightarrow (a_2, a_3, a_4, \ldots, a_{n+2}, \ldots) \in S$ is clearly a bijection $T_3 \rightarrow S$, implying that $|T_3| = |S|$.

   c) $T_2 = \{\alpha \in S \mid \alpha(n) = 1$ and $\alpha(n + 1) = 0$ for no $n \geq 0\}$
      Countable
      Each sequence of T_2 is of the format 0000….00001111… or 0000…
      Consider the bijective map $f : T_2 \rightarrow \mathbb{N}$ such that $f(0000 \ldots\ldots) = 1$

and for all other sequences $\alpha \in T_2, f(\alpha) = n + 2$, where n is the number of zeros.