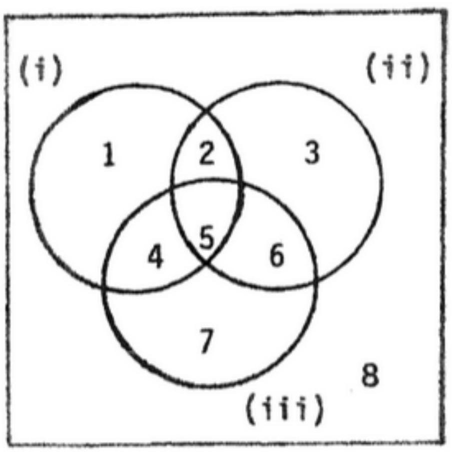
1.

Suppose that $(A-B) \subseteq C$ and $x \in A - C$. Then $x \in A$ but $x \notin C$. If $x \notin B$, then $[x \in A \wedge x \notin B] \Longrightarrow x \in (A-B) \subseteq C$. So now we have $x \notin C$ and $x \in C$. This contradiction gives us $x \in B$, so $(A-C) \subseteq B$.
Conversely, if $(A-C) \subseteq B$, let $y \in A - B$. Then $y \in A$ but $y \notin B$. If $y \notin C$, then $[y \in A \wedge y \notin C] \Longrightarrow y \in (A-C) \subseteq B$. This contradiction, i.e., $y \notin B$ and $y \in B$, yields $y \in C$, so $(A-B) \subseteq C$.

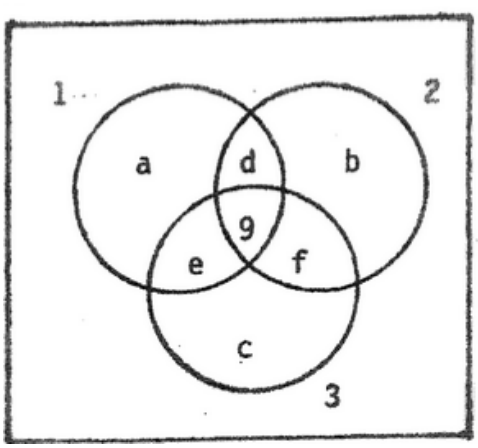
2.



For the given figure let circles (i), (ii), and (iii) denote the subset of assignments where no one is working on experiments 1,2,3, respectively. For each assistant there are seven possibilities: the seven nonempty subsets of {1,2,3}. So there are 7^{15} possible assignments. To determine the number of assignments in region 8 we need to determine the number of assignments in the union of the three subsets. Region 5 has 0 elements, while regions 2,4,6 each contain 1 element (e.g., for region 2, if all assistants are assigned only to experiment 3 then this is the one way that everyone is working on an experiment, but no one is working on experiments 1 and 2).

In each of regions 1,3,7 there are $3^{15}-2$ elements (e.g., for regions 1,2,4,5 there are 3 cases to consider where no one is working on experiment 1 – for each assistant can be working on only experiment 2 or only experiment 3 or both experiments 2,3). The number of assignments where at least one person is working on every experiment is $7^{15}-3[3^{15}-2]-3$.

3.



Consider the Venn diagram shown on the left. From the information given we know that

(i) $a+b+c+d+e+f = 21-9 = 12$;
(ii) $b+c+f=5$;
(iii) $a+c+e=7$; and
(iv) $a+b+d=6$.

Adding equations (ii), (iii) and (iv) we find that $2(a+b+c)+(d+e+f)=18$, so $12=(a+b+c)+[18-2(a+b+c)]$, and the number of students who answered exactly one question is $a+b+c=6$.

4.

$x^2 - 7x \leq -12 \Rightarrow x^2 - 7x + 12 \leq 0 \Rightarrow (x-3)(x-4) \leq 0 \Rightarrow [(x-3) \leq 0$ and $(x-4) \geq 0]$ or $[(x-3) \geq 0$ and $(x-4) \leq 0] \Rightarrow [x \leq 3$ and $x \geq 4]$ or $[x \geq 3$ and $x \leq 4] \Rightarrow 3 \leq x \leq 4$, so $A = \{x | 3 \leq x \leq 4\} = [3, 4]$.

$x^2 - x \leq 6 \Rightarrow x^2 - x - 6 \leq 0 \Rightarrow (x-3)(x+2) \leq 0 \Rightarrow [(x-3) \leq 0$ and $(x+2) \geq 0]$ or $[(x-3) \geq 0$ and $(x+2) \leq 0] \Rightarrow [x \leq 3$ and $x \geq -2]$ or $[x \geq 3$ and $x \geq -2] \Rightarrow -2 \leq x \leq 3$, so $B = \{x | -2 \leq x \leq 3\} = [-2, 3]$.

Consequently, $A \cap B = \{3\}$ and $A \cup B = [-2, 4]$.

5.

(a) Assume that $A \times B \subseteq C \times D$ and let $a \in A$ and $b \in B$. Then $(a, b) \in A \times B$, and since $A \times B \subseteq C \times D$ we have $(a, b) \in C \times D$. But $(a, b) \in C \times D \Rightarrow a \in C$ and $b \in D$. Hence, $a \in A \Rightarrow a \in C$, so $A \subseteq C$, and $b \in B \Rightarrow b \in D$, so $B \subseteq D$.
Conversely, suppose that $A \subseteq C$ and $B \subseteq D$, and that $(x, y) \in A \times B$. Then $(x, y) \in A \times B \Rightarrow x \in A$ and $y \in B \Rightarrow x \in C$ (since $A \subseteq C$) and $y \in D$ (since $B \subseteq D$) $\Rightarrow (x, y) \in C \times D$. Consequently, $A \times B \subseteq C \times D$.

(b) If any of the sets A, B, C, D is empty we still find that

$$[(A \subseteq C) \wedge (B \subseteq D)] \Rightarrow [A \times B \subseteq C \times D].$$

However, the converse need not hold. For example, let $A = \emptyset$, $B = \{1, 2\}$, $C = \{1, 2\}$ and $D = \{1\}$. Then $A \times B = \emptyset$ — if not, there exists an ordered pair (x, y) in $A \times B$, and this means that the empty set A contains an element x. And so $A \times B = \emptyset \subseteq C \times D$ — but $B = \{1, 2\} \not\subseteq \{1\} = D$.

6.

(a) $A \cap B = \{(x, y) | y = 2x + 1$ and $y = 3x\}$
$2x + 1 = 3x \Rightarrow x = 1$
So $A \cap B = \{(1, 3)\}$.
(b) $B \cap C = \{(x, y) | y = 3x$ and $y = x - 7\}$
$3x = x - 7 \Rightarrow 2x = -7$, so $x = -7/2$.
Consequently, $B \cap C = \{(-7/2, 3(-7/2)\} = \{(-7/2, -21/2)\}$.
(c) $\overline{\overline{A} \cup \overline{C}} = \overline{\overline{A}} \cap \overline{\overline{C}} = A \cap C = \{(x, y) | y = 2x + 1$ and $y = x - 7\}$
Now $2x + 1 = x - 7 \Rightarrow x = -8$, and so $A \cap C = \{(-8, -15)\}$.
(d) We know that $\overline{B} \cup \overline{C} = \overline{B \cap C}$, and since $B \cap C = \{(-7/2, -21/2)\}$ we have $\overline{B} \cup \overline{C} = \mathbf{R}^2 - \{(-7/2, -21/2)\} = \{(x, y) | x \neq -7/2$ or $y \neq -21/2\}$.

7.

a) Proof (i): If $a \in \mathbf{Z}^+$, then $\lceil a \rceil = a$ and $\lfloor \lceil a \rceil / a \rfloor = \lfloor 1 \rfloor = 1$. If $a \notin \mathbf{Z}^+$, write $a = n + c$, where $n \in \mathbf{Z}^+$ and $0 < c < 1$. Then $\lceil a \rceil / a = (n+1)/(n+c) = 1 + (1-c)/(n+c)$, where $0 < (1-c)/(n+c) < 1$. Hence $\lfloor \lceil a \rceil / a \rfloor = \lfloor 1 + (1-c)/(n+c) \rfloor = 1$.

Proof (ii): For $a \in \mathbf{Z}^+$, $\lfloor a \rfloor = a$ and $\lceil \lfloor a \rfloor / a \rceil = \lceil 1 \rceil = 1$. When $a \notin \mathbf{Z}^+$, let $a = n + c$, where $n \in \mathbf{Z}^+$ and $0 < c < 1$. Then $\lfloor a \rfloor / a = n/(n+c) = 1 - [c/(n+c)]$, where $0 < c/(n+c) < 1$. Consequently $\lceil \lfloor a \rfloor / a \rceil = \lceil 1 - (c/(n+c)) \rceil = 1$.

8.

(a) $\ldots \cup (-7/3, -2] \cup (-4/3, -1] \cup (-1/3, 0] \cup (2/3, 1] \cup (5/3, 2] \cup \ldots = \bigcup_{m \in \mathbf{Z}^+} (m - 1/3, m]$

(b) $\ldots \cup ((-2n-1)/n, -2] \cup ((-n-1)/n, -1] \cup (-1/n, 0] \cup ((n-1)/n, 1] \cup ((2n-1)/n, 2] \cup \ldots$

$= \bigcup_{m \in \mathbf{Z}^+} (m - 1/n, m]$

9.

(a) Suppose that $x_1, x_2 \in \mathbf{Z}$ and $f(x_1) = f(x_2)$. Then either $f(x_1)$, $f(x_2)$ are both even or they are both odd. If they are both even, then $f(x_1) = f(x_2) \Rightarrow -2x_1 = -2x_2 \Rightarrow x_1 = x_2$. Otherwise, $f(x_1)$, $f(x_2)$ are both odd and $f(x_1) = f(x_2) \Rightarrow 2x_1 - 1 = 2x_2 - 1 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Consequently, the function f is one-to-one.

In order to prove that f is an onto function let $n \in \mathbf{N}$. If n is even, then $(-n/2) \in \mathbf{Z}$ and $(-n/2) < 0$, and $f(-n/2) = -2(-n/2) = n$. For the case where n is odd we find that $(n+1)/2 \in \mathbf{Z}$ and $(n+1)/2 > 0$, and $f((n+1)/2) = 2[(n+1)/2] - 1 = (n+1) - 1 = n$. Hence f is onto.

11.

Solution [Reflexive] $(a,b)\ \rho\ (a,b)$, since $ab = ba$.

[Symmetric] $(a,b)\ \rho\ (c,d)$ implies $(c,d)\ \rho\ (a,b)$, since $ad = bc \Rightarrow cb = da$.

[Transitive] Let $(a,b)\ \rho\ (c,d)$, that is, $ad = bc$, and let $(c,d)\ \rho\ (e,f)$, that is, $cf = de$. Then we get $ad = bc \Rightarrow adf = bcf \Rightarrow adf = bde \Rightarrow af = be$ (since $d \neq 0$), that is, $(a,b)\ \rho\ (e,f)$.

Equivalence classes of A/ρ are of the form $[(a,b)] = \frac{a}{b} = \left\{\frac{na}{nb} \mid n \in \mathbb{N}\right\}$. A unique representative from each equivalence class can be chosen as $\frac{a}{b}$ with $\gcd(a,b) = 1$.

12.

$S = \{(1,2),(2,1)\}$: This relation is Irreflexive, Symmetric, Not Transitive, Not Reflexive, Not Asymmetric, Not antisymmetric.

Equivalence Relation $\implies$ Symmetric, Transitive, Reflexive.

It is not transitive & Reflexive.

Reflexive closure of $S = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

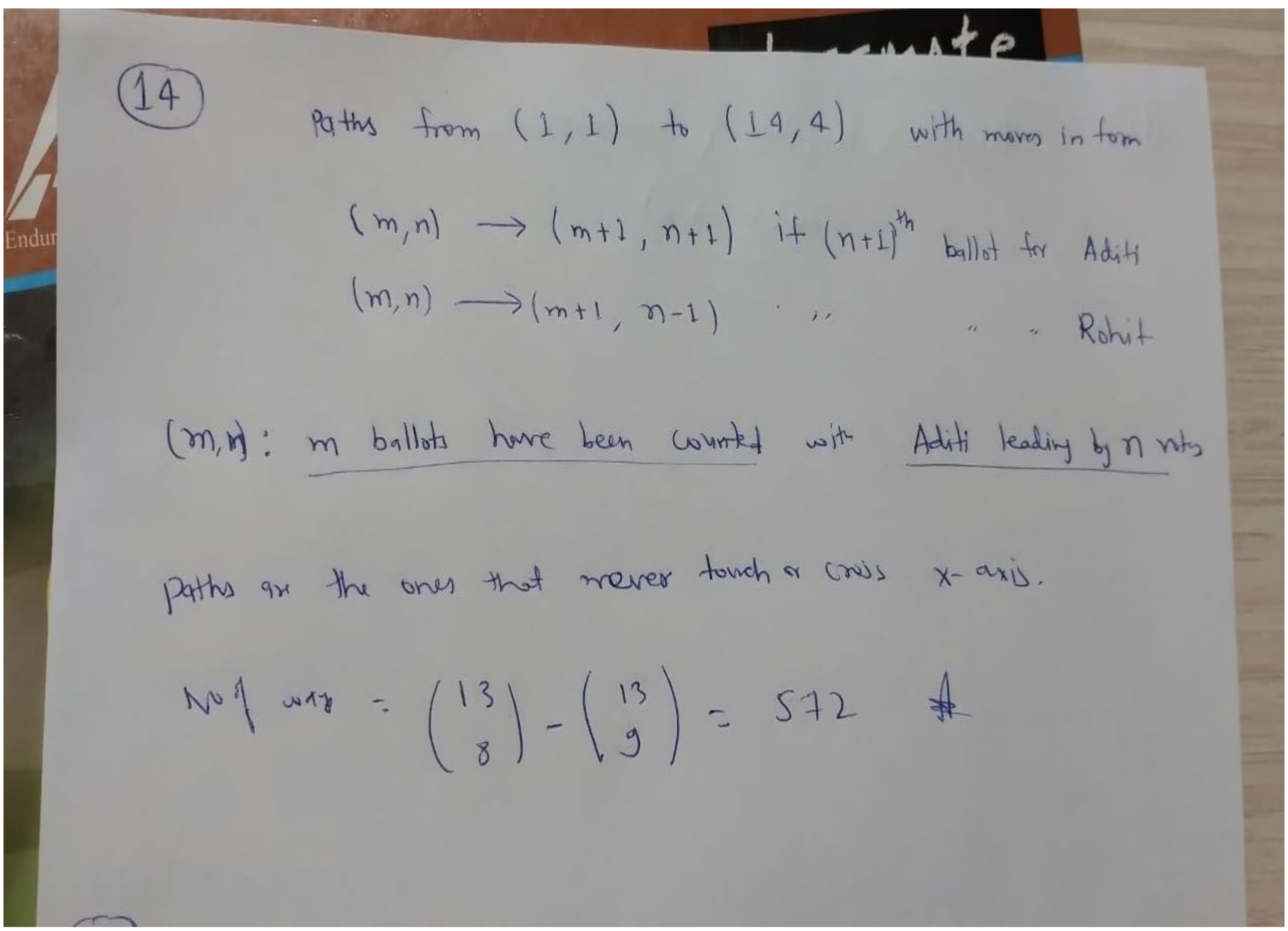
After taking transitive closure relation does not change.

Answer: $S = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

13.

For $1 \le i \le 10$, let x_i be the number of letters typed on day i. Then $x_1 + x_2 + x_3 + \ldots + x_8 + x_9 + x_{10} = 84$, or $x_3 + \ldots + x_8 = 54$. Suppose that $x_1 + x_2 + x_3 < 25, x_2 + x_3 + x_4 < 25, \ldots, x_8 + x_9 + x_{10} < 25$. Then $x_1 + 2x_2 + 3(x_3 + \ldots + x_8) + 2x_9 + x_{10} < 8(25) = 200$, or $3(x_3 + \ldots + x_8) < 160$. Consequently, $54 = x_3 + \ldots + x_8 < (160)/3 = 53\ 1/3$.

14.


(14)

Paths from (1,1) to (14,4) with moves in form

$(m,n) \longrightarrow (m+1, n+1)$ if $(n+1)^{th}$ ballot for Aditi

$(m,n) \longrightarrow (m+1, n-1)$ " " " Rohit

(m,n) : m ballots have been counted with Aditi leading by n votes

Paths are the ones that never touch or cross x-axis.

No of ways $= \binom{13}{8} - \binom{13}{9} = 572$

15.

(15)

(a)

p: Shreya get supervisor position

q: Shreya works hard

r: Shreya gets a raise

s: Shreya buys a new car.

$$\left.\begin{array}{l} (p \wedge q) \rightarrow r \\ \sim s \\ r \rightarrow s \end{array}\right\} \Rightarrow \sim p \vee \sim q$$

(b)

p: There is chance of rain

q: priti's red headband is missing

r: priti does not trim her lawn

s: the temperature is over 40°C

$$\left.\begin{array}{l} (p \vee q) \rightarrow r \\ s \rightarrow \sim p \\ s \wedge \sim q \end{array}\right\} \not\Rightarrow \sim r$$

16.

Proof by Contradiction

(1)	$\neg(p \rightarrow s)$	Premise (Negation of Conclusion)
(2)	$p \wedge \neg s$	Step (1), $(p \rightarrow s) \Longleftrightarrow \neg p \vee s$, DeMorgan's Laws, and the Law of Double Negation
(3)	p	Step (2) and the Rule of Conjunctive Simplification
(4)	$p \rightarrow q$	Premise
(5)	q	Steps (3), (4), and the Rule of Detachment
(6)	r	Premise
(7)	$q \wedge r$	Steps (5), (6), and the Rule of Conjunction
(8)	$(q \wedge r) \rightarrow s$	Premise
(9)	s	Steps (7), (8), and the Rule of Detachment
(10)	$\neg s$	Step (2) and the Rule of Conjunctive Simplification
(11)	$s \wedge \neg s \ (\Longleftrightarrow F_0)$	Steps (9), (10), and the Rule of Conjunction
(12)	$\therefore p \rightarrow s$	Steps (1), (11), and the Method of Proof by Contradiction

17.

The typical palindrome under study here has the form $aba = 100a + 10b + a = 101a + 10b$, where $1 \leq a \leq 9$ and $0 \leq b \leq 9$. With nine choices for a and ten for b, it follows from the rule of product that there are 90 such three-digit palindromes. Their sum is

$$\sum_{a=1}^{9}\left(\sum_{b=0}^{9} aba\right) = \sum_{a=1}^{9}\sum_{b=0}^{9} aba = \sum_{a=1}^{9}\sum_{b=0}^{9}(101a + 10b)$$

$$= \sum_{a=1}^{9}\left[10(101a) + 10\sum_{b=0}^{9} b\right] = \sum_{a=1}^{9}\left[10(101a) + 10\sum_{b=1}^{9} b\right]$$

$$= \sum_{a=1}^{9}\left[1010a + \frac{10(9 \cdot 10)}{2}\right] = \sum_{a=1}^{9}(1010a + 450)$$

$$= 1010\sum_{a=1}^{9} a + 9(450)$$

$$= \frac{1010(9 \cdot 10)}{2} + 4050 = 49{,}500.$$

18.

Proof: (By the Alternative Form of the Principle of Mathematical Induction)
The result holds for $n = 0$ and $n = 1$ because
$(n = 0)\quad 5F_{0+2} = 5F_2 = 5(1) = 5 = 7 - 2 = L_4 - L_0 = L_{0+4} - L_0$; and
$(n = 1)\quad 5F_{1+2} = 5F_3 = 5(2) = 10 = 11 - 1 = L_5 - L_1 = L_{1+4} - L_1$.
This establishes the basis step for the proof.
Next we assume the induction hypothesis — that is, that for some $k\ (\geq 1)$, $5F_{n+2} = L_{n+4} - L_n$ for all $n = 0, 1, 2, \ldots, k-1, k$. It then follows that for $n = k+1$,

$5F_{(k+1)+2} = 5F_{k+3} = 5(F_{k+2} + F_{k+1}) = 5(F_{k+2} + F_{(k-1)+2})$
$= 5F_{k+2} + 5F_{(k-1)+2} = (L_{k+4} - L_k) + (L_{(k-1)+4} - L_{k-1}) = (L_{k+4} - L_k) + (L_{k+3} - L_{k-1})$
$= (L_{k+4} + L_{k+3}) - (L_k + L_{k-1}) = L_{k+5} - L_{k+1} = L_{(k+1)+4} - L_{k+1}$ — where we have used the recursive definitions of the Fibonacci numbers and Lucas numbers to establish the second and eighth equalities.
It then follows by the Alternative Form of the Principle of Mathematical Induction that

$$\forall n \in \mathbf{N}\ \ 5F_{n+2} = L_{n+4} - L_n.$$

19.

a.

$$P \rightarrow \exists x(student(x) \wedge gohome(x) \wedge \forall y(y \neq x \rightarrow solvedm(y)))$$

b.

$$\forall x(residenthostel(x)) \wedge \exists x(dayscholars(x)) \wedge \exists x(dayscholar(x) \wedge \exists y(hostelroom(y) \wedge allotment(x, y)))$$

c.

$$\forall x(\neg sleepat10(x) \rightarrow \neg getup(x)) \wedge \exists x(sleepat10(x) \wedge getup(x)) \wedge \forall x(hostel(x) \rightarrow \exists y(student(y) \wedge (sleepat10(y) \oplus \neg sleep(y))))$$

1.

(a) Prove that ρ is a equivalence relation on $\mathbb{N}$.

Solution [Reflexive] a has the same set of prime divisors as itself, that is, ρ is reflexive.

[Symmetric] If a has the same set of prime divisors as b, then b too has the same set of prime divisors as a, that is, ρ is symmetric.

[Transitive] If a and b have the same set of prime divisors, and b and c have the same set of prime divisors, then a and c too have the same set of prime divisors, that is, ρ is transitive.

(b) Find a unique representative from each equivalence class of ρ.

Solution A non-zero integer is called *square-free* if it is not divisible by the square of a prime number. Each equivalence class of ρ contains a unique square-free integer, and these unique square-free integers are different in distinct equivalence classes. To see why, let $a \in \mathbb{N}$ have the prime factorization $a = p_1^{e_1} \cdots p_t^{e_t}$ with $t \geqslant 0$, with pairwise distinct primes $p_1, \ldots, p_t$ and with each $e_i > 0$. But then $[a] = [p_1 \cdots p_t]$. Moreover, two different square-free integers have different sets of prime divisors. So we can take square-free integers as the representatives of the equivalence classes.

2.

Take $A = \mathbb{Q}$ under the standard $\leqslant$ on rational numbers. Also take $S = \{x \in \mathbb{Q} \mid x^2 > 2\}$. Every rational number $< \sqrt{2}$ is a lower bound on S. Since $\sqrt{2}$ is irrational, $glb(S)$ does not exist.

Another example: Take A to be the set of all irrational numbers between 1 and 5, and S to be the set of all irrational numbers between 2 and 3.

A simpler (but synthetic) example: Take $A = \{a, b, c, d\}$ and the relation on A as,

$$\rho = \{(a,a),(a,c),(a,d),(b,b),(b,c),(b,d)(c,c),(d,d)\}$$

The subset $S = \{c, d\}$ of A has two lower bounds a and b, but these bounds are not comparable to one another.

3.

Solution Let S be a non-empty finite subset of A with $|S| = n$. We prove by induction on n that $lub(S)$ exists. A proof for the existence of $glb(S)$ proceeds analogously.

Since $S \neq \emptyset$, we have $n \geqslant 1$. For $n = 1, 2$, the assertion about the existence of $lub(S)$ is obvious. So take $n \geqslant 3$, and assume that every $(n-1)$- and $(n-2)$-element subset of A has a least upper bound. Take $S = \{a_1, a_2, \ldots, a_n\} \subseteq A$. Since A is a lattice, $b = lub(a_{n-1}, a_n)$ exists. Let $T = \{a_1, a_2, \ldots, a_{n-2}, b\}$. By the induction hypothesis, T has a least upper bound (T has size $n-1$ or $n-2$).

Let U_S (resp. U_T) be the set of all upper bounds of S (resp. T). We first claim that $U_s = U_T$. For the proof, first take $u \in U_S$. Then $a_i \preceq u$ for all $i = 1, 2, \ldots, n$. In particular, $a_{n-1} \preceq u$ and $a_n \preceq u$. Since $b = lub(a_{n-1}, a_n)$, we have $b \preceq u$, that is, $u \in U_T$. Conversely, if $u \in U_T$, then $a_i \preceq u$ for all $i = 1, 2, \ldots, n-2$, and $b \preceq u$. Since b is an upper bound of both a_{n-1} and a_n, we also have $a_{n-1} \preceq b$ and $a_n \preceq b$. By transitivity, $a_{n-1} \preceq u$ and $a_n \preceq u$. Thus $u \in U_S$.

Since T has a least upper bound, U_T is non-empty and contains the unique minimum element $lub(T)$. Since $U_S = U_T$, the same conclusions apply to S as well. It therefore follows that $lub(S) = lub(T)$.

(**Remark:** The above result applies only to finite subsets of A. Infinite subsets may have no least upper bounds and/or no greatest lower bounds. For example, consider the divisibility lattice on $\mathbb{N}$. The lcm of any finite (and non-zero) number of elements exists, but the lcm of an infinite number of (distinct) elements does not exist.)