CS21201 Discrete Structures

Solutions to Practice Problems

Recursive Constructions
Loop Invariance
Pigeonhole Principle

1. If n<=0, function returns -1.
   Let n = $2^k l$, where k is a non-negative integer and l is an odd positive integer.
   If k=0, function returns 0.
   If k>0, n+1 is odd, n*(n+1)/2 can be written as $2^{k-1}m$, where m is odd, and the Function recursively adds 1. Therefore, the function returns k.
   Hence, the function terminates for n > 1.

2. *Solution* The loop of the given function maintains two invariance properties.

   (1) $\gcd(a_1, a_2) = \gcd(x, y)$.

   (2) There exist *integers* v_1, v_2 such that $a_1 = u_1x + v_1y$ and $a_2 = u_2x + v_2y$.

   Initially, $a_1 = x$ and $a_2 = y$, so Invariance (1) is true. It is maintained because $\gcd(a_1, a_2) = \gcd(a_1 - a_2, a_2) = \gcd(a_1, a_2 - a_1)$.

   Invariance (2) is trickier. Given $a_1, a_2, u_1, u_2 \in \mathbb{Z}$, we can always find *unique* values for $v_1 = (a_1 - u_1x)/y$ and $v_2 = (a_2 - u_2x)/y$. What is important to establish in this context is that the way a_1, a_2, u_1, u_2 are updated will ensure that v_1, v_2 will always have integer values.

   Initially, $v_1 = (a_1 - u_1x)/y = (x - x)/y = 0$ and $v_2 = (a_2 - u_2x)/y = (y - 0)/y = 1$ are integers. Assume that $a_1 > a_2$ in some iteration (the other case can be symmetrically handled). At the beginning of the loop, we have integer values for v_1, v_2 such that Invariance (2) holds, that is,

   $$\begin{aligned} a_1 &= u_1x + v_1y, \\ a_2 &= u_2x + v_2y. \end{aligned}$$

   But then,

   $$a_1 - a_2 = (u_1 - u_2)x + (v_1 - v_2)y.$$

   We update a_1 to $a_1 - a_2$, and u_1 to $u_1 - u_2$ in this case ($a_1 > a_2$). If we also update v_1 to $v_1 - v_2$ (and keep v_2 unchanged), then v_1, v_2 will continue to remain integers, and Invariance (2) will continue to hold.

   The loop must terminate, because $\max(a_1, a_2)$ strictly decreases in each iteration. At the end of the loop, we have $a_1 = a_2$, so by Invariance (1), $\gcd(x, y) = \gcd(a_1, a_2) = a_1 = a_2$. Moreover, by Invariance (2), we have $\gcd(x, y) = a_1 = u_1x + v_1y$ for some integer v_1. This v_1 is computed as $(a_1 - u_1x)/y$.

3. The basis step requires that we show that this formula holds when $(m, n) = (0, 0)$. The inductive step requires that we show that if the formula holds for all pairs smaller than (m, n) in the lexicographic ordering of $\mathbf{N} \times \mathbf{N}$, then it also holds for (m, n). For the basis step we have $a_{0,0} = 0 = 0 + 0$. For the inductive step, assume that $a_{m',n'} = m' + n'$ whenever (m', n') is less than (m, n) in the lexicographic ordering of $\mathbf{N} \times \mathbf{N}$. By the recursive definition, if $n = 0$ then $a_{m,n} = a_{m-1,n} + 1$; since $(m - 1, n)$ is smaller than (m, n), the inductive hypothesis tells us that $a_{m-1,n} = m - 1 + n$, so $a_{m,n} = m - 1 + n + 1 = m + n$, as desired. Now suppose that $n > 0$, so that $a_{m,n} = a_{m,n-1} + 1$. Again we have $a_{m,n-1} = m + n - 1$, so $a_{m,n} = m + n - 1 + 1 = m + n$, and the proof is complete.

4. **a)** Since we are working with positive integers, the smallest pair in which the sum of the coordinates is even is $(1, 1)$. So our basis step is $(1, 1) \in S$. If we start with a point for which the sum of the coordinates is even and want to maintain this parity, then we can add 2 to the first coordinate, or add 2 to the second coordinate, or add 1 to each coordinate. Thus our recursive step is that if $(a, b) \in S$, then $(a + 2, b) \in S$, $(a, b + 2) \in S$, and $(a + 1, b + 1) \in S$. To prove that our definition works, we note first that $(1, 1)$ has an even sum of coordinates, and if (a, b) has an even sum of coordinates, then so do $(a + 2, b)$, $(a, b + 2)$, and $(a + 1, b + 1)$, since we added 2 to the sum of the coordinates in each case. Conversely, we must show that if $a + b$ is even, then $(a, b) \in S$ by our definition. We do this by induction on the sum of the coordinates. If the sum is 2, then $(a, b) = (1, 1)$, and the basis step put (a, b) into S. Otherwise the sum is at least 4, and at least one of $(a - 2, b)$, $(a, b - 2)$, and $(a - 1, b - 1)$ must have positive integer coordinates whose sum is an even number smaller than $a + b$, and therefore must be in S by our definition. Then one application of the recursive step shows that $(a, b) \in S$ by our definition.

b) Since we are working with positive integers, the smallest pairs in which there is an odd coordinate are $(1, 1)$, $(1, 2)$, and $(2, 1)$. So our basis step is that these three points are in S. If we start with a point for which a coordinate is odd and want to maintain this parity, then we can add 2 to that coordinate. Thus our recursive step is that if $(a, b) \in S$, then $(a + 2, b) \in S$ and $(a, b + 2) \in S$. To prove that our definition works, we note first that $(1, 1)$, $(1, 2)$, and $(2, 1)$ all have an odd coordinate, and if (a, b) has an odd coordinate, then so do $(a + 2, b)$ and $(a, b + 2)$, since adding 2 does not change the parity. Conversely (and this is the harder part), we must show that if (a, b) has at least one odd coordinate, then $(a, b) \in S$ by our definition. We do this by induction on the sum of the coordinates. If $(a, b) = (1, 1)$ or $(a, b) = (1, 2)$ or $(a, b) = (2, 1)$, then the basis step put (a, b) into S. Otherwise either a or b is at least 3, so at least one of $(a - 2, b)$ and $(a, b - 2)$ must have positive integer coordinates whose sum is smaller than $a + b$, and therefore must be in S by our definition, since we haven't changed the parities. Then one application of the recursive step shows that $(a, b) \in S$ by our definition.

c) We use two basis steps here, $(1, 6) \in S$ and $(2, 3) \in S$. If we want to maintain the parity of $a + b$ and the fact that b is a multiple of 3, then we can add 2 to a (leaving b alone), or we can add 6 to b (leaving a alone). So our recursive step is that if $(a, b) \in S$, then $(a + 2, b) \in S$ and $(a, b + 6) \in S$. To prove that our definition works, we note first that $(1, 6)$ and $(2, 3)$ satisfy the condition, and if (a, b) satisfies the condition, then so do $(a + 2, b)$ and $(a, b + 6)$, since adding 2 or 6 does not change the parity of the sum, and adding 6 maintains divisibility by 3. Conversely (and this is the harder part), we must show that if (a, b) satisfies the condition, then $(a, b) \in S$ by our definition. We do this by induction on the sum of the coordinates. The smallest sums of coordinates satisfying the condition are 5 and 7, and the only points are $(1, 6)$, which the basis step put into S, $(2, 3)$, which the basis step put into S, and $(4, 3) = (2 + 2, 3)$, which is in S by one application of our recursive definition. For a sum greater than 7, either $a \geq 3$, or $a \leq 2$ and $b \geq 9$ (since $2 + 6$ is not odd). This implies that either $(a - 2, b)$ or $(a, b - 6)$ must have positive integer coordinates whose sum is smaller than $a + b$ and satisfy the condition for being in S, and hence are in S by our definition. Then one application of the recursive step shows that $(a, b) \in S$ by our definition.

5.

We track the number of "inversions": pairs {i, j} that occur in the wrong order in the list (if i < j and i is after j in the list, or vice versa). Swapping two elements of the list adds an odd number to the number of inversions. If we swap a and b, then:

- If x is before both of them in the list, or after both of them, then the number of inversions with x is unaffected.
- If x is between them in the list, then both pairs {x, a} and {x, b} change their state of being an inversion or not.
- Finally, {a, b} changes its state.

After an odd number of swaps, the number of inversions has different parity than the start, if the initial number of inversions are odd then final is even and vise-versa. so we cannot be in the initial state.

6. gcd(a,b) is the invariant as gcd(a,b) = gcd(a-b,b) = gcd(a+b,b) = gcd(b,a).

a. S contains (20,23) as gcd(20,23) = gcd(1,2) = 1
b. S does not contain (357, 819) as gcd(357, 819) = 21 ≠ gcd(1,2).
c. gcd(a,b) as already shown.

7. From {k, 2k+1, 3k}, we get 6k+1 ⇔ 12k+3 ⇔ 4k+1 ⇔ 2k ∈ S. This shows that, k ∈ S if and only if 2k and 2k+1 ∈ S, or in the other way, k ∈ S if and only if floor(k/2) ∈ S. Recursively, we get
2023 ∈ S ⇔ 1011 ∈ S ⇔ 505 ∈ S ⇔ 252 ∈ S ⇔ 126 ∈ S ⇔ 63 ∈ S ⇔ 31 ∈ S ⇔ 15 ∈ S ⇔ 7 ∈ S ⇔ 3 ∈ S ⇔ 1 ∈ S.

8.

Solution Consider a person A and the 9 other people in the room. First case, suppose A has at least 4 acquaintances then if any of those acquaintances know each other, we have 3 mutual friends. If none of those acquaintances know each other, we have at least four mutual strangers, namely A's acquaintances. Consider the other case where A has more than 5 strangers (less than 4 acquaintances). Then, consider a person B in the set of A's strangers. B has at least 3 acquaintances or 3 strangers. Suppose B has at least 3 acquaintances, then none of these acquaintances know each other, otherwise we have 3 mutual acquaintances. But since A does not know any of these people, B's 3 acquaintances and A form a group of 4 mutual strangers. On the other hand, if B has at least 3 strangers, then if these strangers all know each other, we would have 3 mutual acquaitances. Thus, suppose there are at least 2 people of B's strangers that do not know each other. But then we would have these 2 people who don't know each other, B, and A as a group of mutual strangers. In all cases, we get either three mutual acquaintances or four mutual strangers, as desired.

9.

Solution The sum of the elements of a subset of $\{1,2,3,\ldots,n^2\}$ of size less than n is $< n^3$. The chosen collection has $2^n - 1$ non-empty subsets. For $n \geqslant 10$, we have $2^n - 1 > n^3$, so there must exist two different non-empty subsets A and B of the chosen numbers such that $\sum_{a \in A} a = \sum_{b \in B} b$. If A and B are not disjoint, take $A - (A \cap B)$ and $B - (A \cap B)$ as A and B.

10. **Solution** To do this, find a four-member set that has at least one element in common with the six sets representing the cards. If any one number appears on three cards this is trivial, pick that one, and one from each other set. If this is not the case, there must be 8 numbers appearing in two sets and 8 appearing in one. Pick a number appearing in two sets, then the union of those two sets is at most 7 distinct numbers. Thus another number can picked that is in two different sets, and then pick one from each of the 2 remaining sets.

11. **Solution** Pick two points A, B of the same color. Let C be the midpoint of AB, and position D, E such that C is also the midpoint of DE and $DA = AB = BE$. If C, D, E are the same color then we're done; if not, then at least one of them is the same color as A, B and forms a trio with A, B.

12. **Solution**. The pigeonholes will be the n sets {1, 2} , {3, 4} , {5, 6} , . . . , {2n − 1, 2n}. The pigeons will be the n + 1 selected numbers. A pigeon is assigned to the unique pigeon hole of which it is a member. By the Pigeonhole Principle, two pigeons must be assigned to some hole, and these are the two consecutive numbers required. Notice that we've actually shown a bit more: there will be two consecutive numbers with the smaller being odd.

CS21201 Discrete Structures
Solutions to Tutorial

Recursive Constructions
Loop Invariance
Pigeonhole Principle

1. All the marbles cannot be of the same color.
   Let us denote the marbles at one stage as (a,b,c) for red, green and blue marbles. It will be transformed into (a+2,b-1,c-1) or (a-1,b+2,c-1) or (a-1,b-1,c+2) after one operation.
   The invariants are a-b (mod 3), b-c (mod 3) and a-c (mod 3).
   For the marbles to be the same color, one invariant needs to be eventually 0 (mod 3).
   However, a=1(mod 3), b=2(mod 3), c=0(mod 3). So the marbles never have same color.

2. **Solution.** *Proof.* Define the *perimeter* of an infected set of students to be the number of edges with infection on exactly one side. Let ν be size (number of edges) in the perimeter.

   We claim that ν is a weakly decreasing variable. This follows because the perimeter changes after a transition only because some squares became newly infected. By the rules above, each newly-infected square is adjacent to at least two previously-infected squares. Thus, for each newly-infected square, at least two edges are removed from the perimeter of the infected region, and at most two edges are added to the perimeter. Therefore, the perimeter of the infected region cannot increase.

   Now if an $n \times n$ grid is completely infected, then the perimeter of the infected region is $4n$. Thus, the whole grid can become infected only if the perimeter is initially at least $4n$. Since each square has perimeter 4, at least n squares must be infected initially for the whole grid to become infected.

3. We use the Pigeonhole Principle. Let A be the set of all sequences of 32 button presses, let B be the set of all configurations, and let f : A → B map each sequence of button presses to the configuration that results. Now:

$$|A| > 4^{32} > 16! > |B|$$

4. *Solution* The total count of 2-subsets of the 65 chosen integers is $\binom{65}{2} = 2080 > 2022$. So we can find two distinct subsets $S = \{a, c\}$ and $T = \{b, d\}$ of the chosen integers such that $(a + c)$ rem $2022 = (b + d)$ rem 2022, that is, $(a - b + c - d)$ rem $2022 = 0$ (where rem means remainder of Euclidean division). We need to show that $S \cap T = \emptyset$. Suppose not. Since S and T are distinct, we must have $|S \cap T| = 1$. Say, $a = b$ (but $c \neq d$). But then, the condition $(a + c)$ rem $2022 = (b + d)$ rem 2022 implies that c rem $2022 = d$ rem 2022. But c and d are chosen in the range $[1, 2022]$, so they must be equal, a contradiction.