Solutions

1. **Base case:** n=1. For this case, $n=1=2^0$. Hence, 1 can be represented as a sum of distinct powers of 2.

   Inductive hypothesis: For $n \in N$, n can be represented as sum of powers of 2.

   Inductive step: Using strong induction, we assume that the inductive hypothesis holds true for all k where $1 \le k \le n$. To prove for n+1, we split the problem into two cases.

   Case 1: n+1 is even. Then (n+1)/2 is an integer, and $1 \le (n+1)/2 \le n$, therefore by inductive hypothesis
   $(n+1)/2 = 2^{a1} + 2^{a2} + ... + 2^{am}$, where a1, a2, ..., am are all distinct.
   Then $n+1 = 2 * (2^{a1} + 2^{a2} + ... + 2^{am}) = 2^{a1+1} + 2^{a2+1} + ... + 2^{am+1}$,
   and (a1+1), (a2+1), ..., (am+1) are all distinct as well.

   Case 2: n+1 is odd. Then n is even, and from the inductive hypothesis,
   $n = 2^{a1} + 2^{a2} + ... + 2^{aj}$, where a1, a2, ..., aj are all distinct and greater than 0.
   $n+1 = 1 + 2^{a1} + 2^{a2} + ... + 2^{aj} = 2^0 + 2^{a1} + 2^{a2} + ... + 2^{aj}$, where 0, a1, a2, ..., aj are all distinct.

   Thus, n+1 can be written as a sum of distinct powers of 2 for both cases.

2. **Base case**: n=1. The first player has no choice but to remove 1 stick and lose.

   Inductive hypothesis: If $n = 4k + 1$, for some $k \in N$, then the second player has a winning strategy; otherwise, the first player has a winning strategy.

   Inductive step: Suppose the theorem is true for numbers 1 through n.
   For the inductive step, there are four cases:

   Case 1: $n + 1 = 4k + 1$. We assume $n + 1 \ge 5$. The first player can remove 1, 2 or 3 sticks. If he\she removes one stick, the remaining number of sticks is $n = 4k$. By the inductive hypothesis, the player who plays at this point has a winning strategy. So the player who played first will lose. Similarly, if the first player removes two sticks, the remaining number is $4(k-1)+3$. Again, the first player loses, by the same reasoning. Similarly, by removing 3 sticks, the first player loses. So, the first player loses for every strategy in this case.

   Case 2: $n + 1 = 4k$. The first player removes 3 sticks: there are now $4(k - 1) + 1$ sticks for the turn of the second player, who loses by the inductive hypothesis.

   Case 3: $n + 1 = 4k + 2$.The first player removes 1 stick: there are now $4k + 1$ sticks for the turn of the second player, who loses by the inductive hypothesis.

Case 4: n + 1 = 4k + 3.The first player removes 2 sticks: there are now 4k + 1 sticks for the turn of the second player, who loses by the inductive hypothesis.

Therefore, the induction hypothesis is proved for all cases.

3.

Solution We proceed by (generalized weak) induction on n with $n_0 = 1$ and $k = 2$.

[Basis] We need two basis cases. For $n = 1$, we have $S_1 = 1^2 = 1$ and $(1+1)! - 1 = 1$. For $n = 2$, $S_2 = 1^2 + 2^2 = 5$ and $(2+1)! - 1 = 6 - 1 = 5$.

[Induction] Assume that $S_{n-1} = n! - 1$ and $S_{n-2} = (n-1)! - 1$ for some $n \geqslant 3$. All non-empty subsets of $\{1, 2, 3, \ldots, n\}$ that do not contain consecutive integers can be classified in three groups.

1. Non-empty subsets of $\{1, 2, 3, \ldots, n-1\}$ that do not contain consecutive integers.
2. A non-empty subset with the desired property that contains n and one or more elements from $\{1, 2, 3, \ldots, n-1\}$. Since these subsets are not allowed to contain consecutive integers, the elements other than n must come from $\{1, 2, 3, \ldots, n-2\}$.
3. The subset $\{n\}$.

By induction, it follows that

$$\begin{aligned} S_n &= S_{n-1} + n^2 S_{n-2} + n^2 \\ &= (n! - 1) + n^2\Big((n-1)! - 1\Big) + n^2 \\ &= n! + n^2 \times (n-1)! - 1 \\ &= (n-1)!(n + n^2) - 1 \\ &= (n+1)! - 1. \end{aligned}$$

4. For proving the statement with strong induction, the lemma “Every simple polygon with at least four sides has an interior diagonal” is required.

Base case: For n=3, the polygon is a triangle, it is triangulated into 3-2=1 triangles.

Induction hypothesis: A simple polygon with n sides, where n is an integer with n ≥ 3, can be triangulated into n − 2 triangles.

Induction step: Using strong induction, we assume that we can triangulate a simple polygon with j sides into j−2 triangles whenever 3 ≤ j ≤ k.
Suppose we have a simple polygon P with k + 1 sides. Because k + 1 ≥ 4, from the lemma P has an interior diagonal ab. Now, ab splits P into two simple polygons Q, with s sides, and R, with t sides. The sides of Q and R are the sides of P, together with the side ab, which is a side of both Q and R. 3 ≤ s ≤ k and 3 ≤ t ≤ k because both Q and R have at least one fewer side than P does. We also get k + 1 = s + t − 2, because both Q and R have the diagonal as a side that is not a part of P.

We now use the inductive hypothesis. Because both 3 ≤ s ≤ k and 3 ≤ t ≤ k, by the inductive hypothesis we can triangulate Q and R into s − 2 and t − 2 triangles, respectively. These triangulations together produce a triangulation of P. (Each diagonal added to triangulate one of these smaller polygons is also a diagonal of P.) Consequently, we can triangulate P into a total of $(s - 2) + (t - 2) = s + t - 4 = (k + 1) - 2$ triangles. This completes the proof by strong induction.