CS21201 Discrete Structures
Solutions to Practice Problems

Predicate Logic

1. Coding of the statements is as under ($x \in$ Creatures)
   S1: $\forall x\ [Lion(x) \rightarrow Fierce(x)]$
   S2: $\exists x\ [Lion(x)\ \wedge\ \neg DrinksC(x)]$

   a. $\exists x\ Fierce(x)$
   This statement is true if there is at least one lion. We cannot say that this directly follows from S1. This statement is **False.**

   b. From S2, we notice that $\exists x\ [Lion(x)\ \wedge\ \neg DrinksC(x)]$. So let that creature be p. We know that p is a lion. From S1, any creature who is a lion is fierce. Therefore p is fierce. By existential generalization, $\exists x\ Fierce(x)$. Hence this statement is **True.**

   c. $\exists x\ [Fierce(x)\ \wedge\ \neg DrinksC(x)]$: Notice that similar to (b), we derive that p is a lion and p does not drink coffee. From S1, all lions are fierce. Therefore p is fierce. This implies that $Fierce(p)\ \wedge\ \neg DrinksC(p)$, by existential generalization, $\exists x\ [Fierce(x)\ \wedge\ \neg DrinksC(x)]$. The statement is **True.**

2. Predicates used are $Respect(x,\ y)$: person x respects person y and $Hire(x,\ y)$: person x hires person y.

   S1 : $\forall x\ (\neg Respect(x,\ x)\ \rightarrow \neg\exists y(Respect(y,\ x))$
   S2 : $\forall x \forall y\ (\neg Respect(x,\ y)\ \rightarrow \neg Hire(x,\ y))\ \equiv\ \forall x \forall y\ (Hire(x,\ y)\ \rightarrow Respect(x,\ y))$
   G : $\forall x\ [(\neg\exists y\ Respect(x,\ y))\ \rightarrow (\neg\exists z\ Hire(z,\ x))]$

   Simplification of G:
   $\forall x\ [(\neg\exists y\ Respect(x,\ y))\ \rightarrow (\neg\exists z\ Hire(z,\ x))]$ (1)
   $\forall x\ [(\exists z\ Hire(z,\ x))\ \rightarrow (\exists y\ Respect(x,\ y))]$ (2) Contrapositive (1)

   Proof by contradiction, assume that $\neg G$ is true.
   $\neg\forall x\ [(\exists z\ Hire(z,\ x))\ \rightarrow (\exists y\ Respect(x,\ y))]$ (3)
   $\exists x\ \neg[\neg(\exists z\ Hire(z,\ x))\ \vee\ (\exists y\ Respect(x,\ y))]$ (4) Properties of $\neg$ and $\rightarrow$
   $\exists x\ [(\exists z\ Hire(z,\ x))\ \wedge\ \neg(\exists y\ Respect(x,\ y))]$ (5) De Morgan's Laws
   Instantiate (5), by $x\ =\ A$ and $z\ =\ B$

$Hire(B, A)$ (6)

$\neg(\exists y\, Respect(A, y))$ (7)

Instantiate S2 by $x = B$ and $z = A$ (8)

$Hire(B, A) \rightarrow Respect(B, A)$

$Respect(B, A)$ (9) Modus Ponens (7, 8)

$\forall x(\exists y\, (Respect(y, x)) \rightarrow Respect(x, x))$ (10) Contrapositive (S1)

Instantiate by $x = A$ and $y = B$

$Respect(B, A) \rightarrow Respect(A, A)$ (11)

$Respect(A, A)$ (12) Modus Ponens(9, 11)

But from (7), $\neg(\exists y\, Respect(A, y)) \Rightarrow \forall y\, \neg Respect(A, y) \Rightarrow \neg Respect(A, A)$

Hence we have a contradiction

3. Predicates:

$a(x)$: Person x belongs to the Alpine Club

$s(x)$: Person x is a skier

$m(x)$: Person x is a mountain climber

$l(x, y)$: Person x likes weather event y

Statements:

S1 : $a(Tony) \wedge a(Mike) \wedge A(John)$

S2 : $\forall x\, [a(x) \rightarrow (s(x) \vee m(x))]$

S3 : $\neg\exists x\, [m(x) \wedge l(x, Rain)]$

S4 : $\forall x[s(x) \rightarrow l(x, Snow)]$

S5 : $\forall y[l(Mike, y) \leftrightarrow \neg l(Tony, y)]$

S6 : $l(Tony, Rain) \wedge l(Tony, Snow)$

Since Tony likes both Rain and Snow and Mike dislikes whatever Tony likes and likes whatever Tony dislikes

Mike does not like Rain and Mike does not like Snow

$\neg l\, (Mike, Rain)$ (1)

$\neg l\, (Mike, Snow)$ (2)

From S4, instantiating $x = Mike$, we get

$s(Mike) \rightarrow l(Mike, Snow)$ (3)

$\neg s(Mike)$ (4) Modus Tollens(2, 3)

From S2, instantiating $x = Mike$, we get

$a(Mike) \rightarrow (s(Mike) \vee m(Mike))$ (5)

$s(Mike) \vee m(Mike)$ (6) Modus Ponens(S1, 5)

$m(Mike)$ (7) (4, 6)

Clearly **Mike** is a Mountain Climber and not a skier, from (4) and (7).

Proof Techniques

1.

Let $P(n)$ be $n! < n^n$.

Base case : We show that $P(n)$ holds for $n = 2$.
$n! = 2! = 2$
$n^n = 2^2 = 4$
$2 < 4 \to n! < n^n$ when $n = 2$.

Induction hypothesis : Assume that $P(n)$ holds for some $n \in \mathbb{N}$. That is, $n! < n^n$ for some $n \in \mathbb{N}$.

Induction step : We show that $P(n+1)$ holds. Consider $(n+1)!$.
$(n+1)!$
$= (n+1)n!$ by definition of factorial
$< (n+1)n^n$ by the induction hypothesis
$< (n+1)(n+1)^n$ by Lemma 1
$< (n+1)^{n+1}$

We have shown that $(n+1)! < (n+1)^{n+1}$, thus $P(n+1)$ holds, completing the induction.

Lemma 1[4]
We show that $n^n < (n+1)^n$ for $n > 1$.

Binomial theorem : $(a+b)^k = a^k + ka(k-1)b + \cdots + b^k$

$(n+1)^n$
$= n^n + n \cdot n^{n-1} + ...$ using the Binomial Theorem
$> n^n$

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[4]This is a higher level of detail than we required for full credit.

2.

Solution [Induction basis] For $n = 4$, we have $2^4 = 16 < 4! = 24 < 2^{4\log_2 4} = 256$.

[Induction] Suppose that $2^n < n! < 2^{n\log_2 n}$ for some $n \geqslant 4$. We then have

$$(n+1)! = (n+1) \times n! > (n+1) \times 2^n > 2 \times 2^n = 2^{n+1},$$

and

$$(n+1)! = (n+1) \times n! < (n+1) \times 2^{n\log_2 n} = (n+1) \times n^n <= (n+1)^{n+1} = 2^{(n+1)\log_2(n+1)}.$$

3.

Solution By the extended gcd, we always have a representation of the form $1 = u(\frac{a}{d}) + v(\frac{b}{d})$ for some integers u, v. Write $u = q(\frac{b}{d}) + r$ with $0 \leqslant r < \frac{b}{d}$ (Euclidean division). We then have $1 = \Big(q(\frac{b}{d}) + r\Big)(\frac{a}{d}) + v(\frac{b}{d}) = r(\frac{a}{d}) + s(\frac{b}{d})$, where $s = v + q(\frac{a}{d})$. If $r = 0$, then $s = \frac{d}{b} \leqslant 1 \leqslant \frac{a}{d}$. If $r > 0$, then $|s| = \frac{d}{b}\Big(r(\frac{a}{d}) - 1\Big) < \Big(r(\frac{d}{b})\Big)\frac{a}{d} < \frac{a}{d}$.

4: Proof very similar to the problem done in the class

5. Hint: Let $n = p1^{e1} p2^{e2} \ldots . pk^{ek}$. If n is not a perfect square, at least one of the e1, e2,...ek will be odd (Note though that more than one such ei's can be odd too). Use this fact to prove.

6.

a.

> **Base Case:** $n = 1$. In this case, we have that $1 + \cdots + 2^n = 1 + 2 = 2^2 - 1$, and the statement is therefore true.
>
> **Inductive Hypothesis:** Suppose that for some $n \in \mathbb{N}$, we have $1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

Inductive Step: Consider

$$\begin{aligned} 1+2+4+\cdots+2^{n+1} &= 1+2+4+\cdots+2^n+2^{n+1} \\ &= (2^{n+1}-1)+2^{n+1} \quad \text{(by the Inductive Hypothesis)} \\ &= 22^{n+1}-1 \\ &= 2^{(n+1)+1}-1. \end{aligned}$$

Therefore, we have that if the statement holds for n, it also holds for $n+1$. By induction, then, the statement holds for all $n \in \mathbb{N}$.

b.

Base Case: $n = 1$. In this case, we have $\sum_{k=1}^{n} k(k+1) = 1(2) = 2 = \frac{1(2)(3)}{3}$, so the result holds.

Inductive Hypothesis: Suppose, for some $n \in \mathbb{N}$, we have $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$.

Inductive Step: Consider the case of $n+1$. In this case, we have

$$\begin{aligned} \sum_{k=1}^{n+1} k(k+1) &= \sum_{k=1}^{n} k(k+1) + (n+1)(n+2) \\ &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \quad \text{(by the Inductive Hypothesis)} \\ &= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3} \\ &= \frac{(n+1)(n+2)(n+3)}{3} \quad \text{(by factoring out } (n+1)(n+2)\text{)}. \end{aligned}$$

Therefore, we have that if the statement holds for n, it also holds for $n+1$.

Hence, by induction, $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{N}$. □

c.

Base Case: $n = 5$. Then $4n = 20 < 32 = 2^n$, and the result is true.

Inductive Hypothesis: Suppose, for some $n \geq 5$, that $4n < 2^n$.

Inductive Step: Consider $n + 1$. We have

$$\begin{aligned} 4(n+1) &= 4n + 4 \\ &< 2^n + 4 \quad \text{(by the Inductive Hypothesis)} \\ &< 2^n + 2^n \quad \text{(since } 2^n > 4 \text{ for all } n \geq 5\text{)} \\ &= 2^{n+1}. \end{aligned}$$

Therefore, if the result holds for n, it also holds for $n + 1$.

Thus, by induction, we have that $4n < 2^n$ for all $n \geq 5$.

CS21201 Discrete Structures

Solutions to Tutorial Problems

Predicate Logic and Proof Techniques

1. Notice that the domain of x and y have been restricted to Candidates and Questions respectively, this is to minimize the predicates to be used. Any other method which involves predicates like $C(x)$: x is a candidate, $Q(y)$: y is a question and $A(x,\ y)$: x answers y is also fine.

Q1

a) $A(x, y)$: Candidate x answers Question y

b) S1 : $\exists y\ \forall x \left[\left(\exists z\ A(x, z) \right) \longrightarrow A(x, y) \right]$

S2 : $\forall x\ \exists y\ A(x, y)$

G : $\exists y\ \forall x\ A(x, y)$

c) **S1**

$\exists y\ \forall x \left[\left(\exists z\ A(x, z) \right) \longrightarrow A(x, y) \right]$

Let $y = P$ (instantiation)

Then,

S1 : $\forall x \left[\left(\exists z\ A(x, z) \right) \longrightarrow A(x, P) \right]$

S2 : $\forall x \left[\exists z\ A(x, z) \right]$

Do universal instantiation for every x [i.e. Every candidate]

S1 : $\exists z\ A(c, z) \longrightarrow A(c, P)$

S2 : $\exists z\ A(c, z)$

$$\frac{\begin{array}{l} \exists z\ A(c, z) \longrightarrow A(c, P) \\ \exists z\ A(c, z) \end{array}}{A(c, P)} \quad \text{[Modus Ponens]}$$

$A(c, P)$

$\therefore\ \forall x\ A(x, P)$ [Universal generalization]

$\therefore\ \exists y\ \forall x\ A(x, y) \equiv G$ [Existential generalization]

2. (a) Bob imitates Alice.

   (b) Alice forces Bob to the situation m = n.

3. (b)

Solution [Base] For $n = 1$, we have $H_1 = 1 \leqslant 1 + 0 = 1 + \ln 1$.

[Induction] Take $n \geqslant 1$, and assume that $H_n \leqslant 1 + \ln n$. Then we have

$$\begin{aligned}
H_{n+1} &= H_n + \frac{1}{n+1} \\
&\leqslant 1 + \ln n + \frac{1}{n+1} \\
&= 1 + \ln(n+1) + \frac{1}{n+1} + (\ln n - \ln(n+1)) \\
&= 1 + \ln(n+1) + \frac{1}{n+1} + \ln\left(\frac{n}{n+1}\right) \\
&= 1 + \ln(n+1) + \frac{1}{n+1} + \ln\left(1 - \frac{1}{n+1}\right) \\
&= 1 + \ln(n+1) + \frac{1}{n+1} - \frac{1}{n+1} - \frac{1}{2}\left(\frac{1}{n+1}\right)^2 - \frac{1}{3}\left(\frac{1}{n+1}\right)^3 - \frac{1}{4}\left(\frac{1}{n+1}\right)^4 - \cdots \\
&= 1 + \ln(n+1) - \left[\frac{1}{2}\left(\frac{1}{n+1}\right)^2 + \frac{1}{3}\left(\frac{1}{n+1}\right)^3 + \frac{1}{4}\left(\frac{1}{n+1}\right)^4 + \cdots\right] \\
&\leqslant 1 + \ln(n+1).
\end{aligned}$$

Since $n \geqslant 1$, we have $0 < \frac{1}{n+1} \leqslant \frac{1}{2} < 1$, and so we can use the above expansion of $\ln\left(1 - \frac{1}{n+1}\right)$.