CS21201 Discrete Structures

Solutions to Practice Problems

Propositional Logic

1. Coding using Propositional Logic is as follows:

   S1 : $\neg S \lor D$
   S2 : $\neg R \lor E$
   S3 : $(S \land \neg R) \lor (\neg S \land R)$
   S4 : $\neg S \lor E$
   S5 : $\neg R \lor D$
   G : $(\neg E \lor \neg D) \land (D \lor E)$

   The goal should be easy enough to derive using the truth table method

2. **Solution I**: Assume that Ronaldo knows Mbappe means that they both know each other. Propositions used are as follows:

   K : Mbappe knows Ronaldo (Ronaldo knows Mbappe)
   HL : Haaland likes the cookies
   MBP : Mbappe was on the pitch
   MEP : Messi was on the pitch

   Messi: $K \land HL$
   Mbappe: $\neg K \land \neg MBP$
   Haaland: $MBP \land MEP$

   Since one and only one of Haaland, Mbappe or Messi ate the cookies. We break the problem down into three cases depending on who ate the cookies:

   a. Messi: Both Mbappe and Haaland are telling the truth. But for that to be true, $\neg MBP$ and MBP have to be true at the same time, **not possible**
   b. Haaland: Both Messi and Mbappe are telling the truth. But for that to be true, K and $\neg K$ have to be true, **not possible**
   c. Mbappe: It can be seen that this is the only possible case. Since we can keep HL = 1 and MEP = 1.

   Solution II: Assume that Ronaldo knows Mbappe but Mbappe does not know Ronaldo. In that case, we design two predicates:
   KMR : Mbappe knows Ronaldo
   KRM : Ronaldo knows Mbappe

Messi: $KRM \wedge HL$
Mbappe: $\neg KMR \wedge \neg MBP$
Haaland: $MBP \wedge MEP$

Since one and only one of Haaland, Mbappe or Messi ate the cookies. We break the problem down into three cases depending on who ate the cookies:

a. Messi: Not possible, see **Solution I**
b. Haaland: Both Messi and Mbappe are telling the truth. But for that to be true, KRM and $\neg KMR$ have to be true (in addition to HL and $\neg MBP$), **possible**
c. Mbappe: Possible, see **Solution I**

Under this assumption, the identity of the thief cannot be ascertained.

3. Denote the $NAND(x)$ function as $N(x)$. You can write $\neg X$ as $N(X, X)$.
   a. $P \rightarrow Q \equiv \neg P \vee Q \equiv \neg(P \wedge \neg Q)$
   $\equiv N(P, \neg Q)$ (Using De Morgan's Theorem)
   $\equiv N(P, N(Q, Q))$

   b. $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$
   $\equiv N(P, N(Q, Q)) \wedge N(Q, N(P, P))$

   To simplify things a bit, choose:
   $X = N(P, N(Q, Q))$ and $Y = N(Q, N(P, P))$

   $P \leftrightarrow Q \equiv X \wedge Y \equiv \neg(\neg X \vee \neg Y)$ (using De Morgan's Theorem)
   $\equiv N(N(X, X), N(Y, Y))$

   c. $P \oplus Q \equiv (P \vee Q) \wedge (\neg Q \vee \neg P)$
   $X = (P \vee Q) \equiv N(\neg P, \neg Q) \equiv N(N(P, P), N(Q, Q))$
   $Y = (\neg P \vee \neg Q) \equiv N(P, Q)$
   Repeat the solution to part (b).

4.
   a. Notice that any formula constructed with $\oplus$ will always be false when all propositions are false. However, consider $\neg A$. If A is false, then $\neg A$ is true. Therefore, there are propositional logic formulae which can never be covered by $\oplus$ and hence $\{\oplus\}$ is not functionally complete.

b. This question is wrong. It can be seen that any combination of $\{\oplus, \wedge\}$ is always false whenever all propositions are false, since both $A \oplus B$ and A $A \wedge B$ give false values when both A and B are false.
Similar to part (a), consider $\neg A$, to continue.

5.

a. It is sufficient to prove that $(p \vee q) \wedge (\neg q \vee r) \wedge \neg(p \vee r)$ is a contradiction:
$(p \vee q) \wedge (\neg q \vee r) \wedge \neg(p \vee r) \equiv (p \vee q) \wedge (\neg q \vee r) \wedge \neg p \wedge \neg r$
By Disjunctive Syllogism,
$(p \vee q) \wedge \neg p \rightarrow q$ and
$(\neg q \vee r) \wedge \neg r \rightarrow \neg q$
Hence Proved

b.

i. This question has been changed, consider the corrected problem as uploaded on the webpage:

$p \rightarrow s \equiv \neg p \vee s$	(1)
$p \vee (q \wedge r)$	(2)
$s \vee (q \wedge r)$	(3) Resolution Rule (2, 3)
$(s \vee q)$	(4) Distribution (3)
$(s \vee r)$	(5) Distribution (3)

Hence proved
Note that in the original problem, the goal could not be derived from the statements.

ii.

$(\neg q \vee r)$	(1)
$(p \vee q \vee t)$	(2)
$(p \vee r \vee t)$	(3) Resolution Rule (2, 3)
$(\neg p \vee s)$	(4)
$(s \vee r \vee t)$	(5) Resolution Rule (3, 4)
$\neg t \vee (s \wedge r)$	(6)
$(s \vee r) \vee (s \wedge r)$	(7) Resolution Rule (5, 6)
$(s \vee r \vee s) \wedge (s \vee r \vee r)$	(8) Distribution (7)
$(s \vee r)$	(9) Simplification

6.

a. Trivial, consider an alternate form of $A \rightarrow B$ as $\neg A \vee B$.
   $Q = \neg(S_1 \wedge S_2 \wedge \ldots \wedge S_n) \vee G$ is a tautology.
   Therefore, $\neg Q$ is a contradiction.
   $\neg Q = \neg(\neg(S_1 \wedge S_2 \wedge \ldots \wedge S_n) \vee G) = (S_1 \wedge S_2 \wedge \ldots \wedge S_n) \wedge \neg G$

b. S1: $(p \rightarrow q) \rightarrow q \equiv \neg(\neg p \vee q) \vee q \equiv (p \wedge \neg q) \vee q \equiv p \vee q$
   S2: $(p \rightarrow p) \rightarrow r \equiv \neg(\neg p \vee p) \vee r \equiv r$
   Hence the goal directly follows from S2.

7. Consider the propositions for the problem:
   WF : Water altar contains the mysteries of fire
   EF : Earth altar contains the mysteries of fire
   EW : Earth altar holds the secrets of water
   AW : Air altar holds the secrets of water
   FE : Fire altar holds the energy of earth
   AE : Air altar holds the energy of earth
   AF : Air altar holds the mysteries of fire
   AA : Air altar holds the strength of air
   EA : Earth altar holds the strength of air
   FW : Fire altar holds the secrets of water
   FA : Fire altar holds the strength of air
   FE : Fire altar holds the energy of earth

   a. $\neg(WF \wedge EF) \equiv \neg WF \vee \neg EF$

   b. $\neg(\neg EW \wedge \neg AW) \equiv EW \vee AW$

   c. $\neg(FE \wedge (\neg AW \wedge \neg AE \wedge \neg AF \wedge \neg AA))$
      $\neg FE \vee \neg(\neg AW \wedge \neg AE \wedge \neg AF \wedge \neg AA) \equiv \neg FE \vee AW \vee AE \vee AF \vee AA$

   d. $\neg(\neg EA \wedge (\neg FW \wedge \neg FE \wedge \neg FF \wedge \neg FA))$
      $EA \vee FW \vee FE \vee FF \vee FA$

From the second statement, it is clear that either the earth altar or the air altar holds the power of water.

Earth altar holds the power of water

Consider $EW = 1, FA = 1, AE = 1, WF = 1$

Air altar holds the power of water

Consider $AW = 1, FA = 1, AE = 1, WF = 1$

Therefore, both the Earth and Air altars can hold the power of water.

8. $(r \to \neg q)$ (1)
   $(p \to q)$ (2)
   $(q \to \neg r)$ (3) Contrapositive (1)
   $(p \to \neg r)$ (4) Transitive Implication
   $(r \to \neg p)$ (5) Contrapositive (4)

   It is clear from the above derivation that → holds. However, ← does not hold. Choose the values of p, q and r as $r = T$, $p = F$ and $q = T$.

9.
   a. $(Smoke \to Fire) \to (\neg Smoke \to \neg Fire)$ (1)
      $\equiv (Smoke \to Fire) \to (Fire \to Smoke)$ (2) Contrapositive (1)

      Valid: No ($Smoke = F, Fire = T$)
      Satisfiable: Yes ($Smoke = T, Fire = T$)

   b. $((Smoke \land Heat) \to Fire) \leftrightarrow ((Smoke \to Fire) \lor (Heat \to Fire))$
      $((Smoke \land Heat) \to Fire) \equiv \neg Smoke \lor \neg Heat \lor Fire$
      $\equiv \neg Smoke \lor Fire \lor \neg Heat \lor Fire \equiv (Smoke \to Fire) \lor (Heat \to Fire)$
      Valid: Yes
      Satisfiable: Yes

   c. $Big \lor Dumb \lor (Big \to Dumb)$
      $\equiv Big \lor Dumb \lor (\neg Big \lor Dumb) \equiv Dumb$

      Valid: No ($Dumb = F$)
      Satisfiable: Yes ($Dumb = T$)

   d. $(Big \land Dumb) \lor \neg Dumb \equiv (Big \lor \neg Dumb) \land (Dumb \lor \neg Dumb)$
      $(Big \lor \neg Dumb)$
      Valid: No ($Big = F, Dumb = T$)
      Satisfiable: Yes (all other truth values)

   e. $(Smoke \to Fire) \to ((Smoke \land Heat) \to Fire)$
      $Heat = T \Rightarrow (Smoke \to Fire) \to (Smoke \to Fire) = T$
      $Heat = F \Rightarrow (Smoke \to Fire) \to T = T$
      Valid: Yes
      Satisfiable: Yes

CS21201 Discrete Structures

Solutions to Tutorial Problems

Propositional Logic

1.

Tutorial – 2
Propositional Logic

Question – 1

(a) Propositions used are as follows:

- p : Swapna wrote the paper
- t : Swapna typed the answers
- h : Swapna has a camera
- c : Swapna completed in time

(b)

$S1$: $p \vee t$
$S2$: $(p \vee \neg h) \rightarrow \neg c$
$S3$: $\neg c$

G : p

(c) $\boxed{p = F,\ c = F,\ h = F,\ t = T}$

S1, S2 and S3 are true but G is false hence $S1 \wedge S2 \wedge S3 \rightarrow G$ is not a Tautology.

2.

Solution The first deduction goes as follows.

$$\frac{\neg(s \vee u)}{\therefore\ \neg s\ ,\ \neg u} \qquad \frac{\neg u \quad t \rightarrow u}{\therefore\ \neg t} \qquad \frac{r \rightarrow (s \vee t) \quad \neg s\ ,\ \neg t}{\therefore\ \neg r} \qquad \frac{(\neg p \vee q) \rightarrow r \quad \neg r}{\therefore\ p \wedge \neg q} \qquad \frac{q \leftrightarrow v \quad \neg q}{\therefore\ \neg v}$$

$$\frac{(v \wedge \neg w) \vee (\neg v \wedge w) \rightarrow \neg p \quad p}{\therefore\ (\neg v \vee w) \wedge (v \vee \neg w)} \qquad \frac{(\neg v \vee w) \wedge (v \vee \neg w) \quad \neg v}{\therefore\ \neg w}$$

3.

a.

Solution We may use the following propositions.

rob :	The murder was done for robbery.
pol :	The murder was a political assassination.
wom :	The murder was for a woman.
tak :	Something was taken from the murderer's place.
imm :	The assassin left immediately after work done.
trc :	The assassin left tracks all over the room.

b.

Solution
(1) $\neg \text{rob} \rightarrow \text{pol} \vee \text{wom}$
(2) $\text{rob} \rightarrow \text{tak}$
(3) $\neg \text{tak}$
(4) $\text{pol} \rightarrow \text{imm}$
(5) trc
(6) $\text{trc} \rightarrow \neg \text{imm}$

c.

Solution

$\text{rob} \rightarrow \text{tak}$
$\neg \text{tak}$
__________ (Modus Tollens)
$\therefore \neg \text{rob}$

$\neg \text{rob}$
$\neg \text{rob} \rightarrow \text{pol} \vee \text{wom}$
__________ (Modus Ponens)
$\therefore \text{pol} \vee \text{wom}$

trc
$\text{trc} \rightarrow \neg \text{imm}$
__________ (Modus Ponens)
$\therefore \neg \text{imm}$

$\neg \text{imm}$
$\text{pol} \rightarrow \text{imm}$
__________ (Modus Tollens)
$\therefore \neg \text{pol}$

$\text{pol} \vee \text{wom}$
$\neg \text{pol}$
__________ (Disjunctive Syllogism)
$\therefore \text{wom}$

d. The murder was done for a woman