CS21201 Discrete Structures
Tutorial Problems

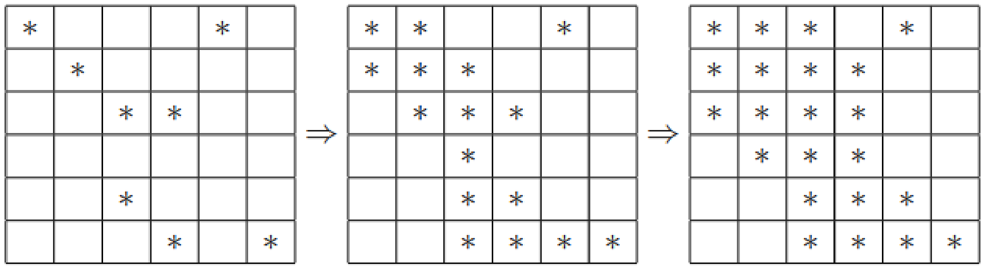
Recursive Constructions
Loop Invariance
PigeonHole Principle

1. A box contains 100 red marbles, 101 green marbles, and 102 blue marbles. You also have an unlimited external store of marbles of each of these colors. You perform the following experiments. As long as the box contains marbles of at least two different colors, repeat: take two marbles, each with a different color from the box to the store, and then move two marbles of the remaining color from the store to the box. You stop when all the marbles in the box are of the same color. Is it possible for all the marbles in the box to have the same color, is the experiment possible? What is this color?

2. An outbreak of beaver flu sometimes infects students in class, which lasts forever. Here is an example of a 6×6 class arrangement with the locations of infected students marked with an asterisk.

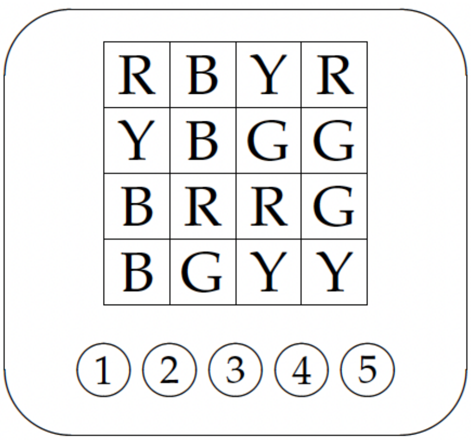
*				*	
	*				
		*	*		
		*			
			*		*

Outbreaks of infection spread rapidly step by step. A student is infected after a step if either
• the student was infected at the previous step (since beaver flu lasts forever).
• the student was adjacent to at least two already-infected students at the previous step.
Here adjacent means the students' individual squares share an edge (front, back, left or right, but not diagonal). Thus, each student is adjacent to 2, 3 or 4 others. In the example, the infection spreads as shown below.



In this example, over the next few time-steps, all the students in class become infected.
Theorem: If fewer than n students among those in an n × n arrangement are initially infected in a flu outbreak, then there will be at least one student who never gets infected in this outbreak, even if students attend all the lectures. Prove this theorem.

3. An electronic toy displays a 4 × 4 grid of colored squares. At all times, four are red, four are green, four are blue, and four are yellow. For example, here is one possible configuration:



Below the display, there are five buttons numbered 1, 2, 3, 4, and 5. The player may press a sequence of buttons; however, the same button can not be pressed twice in a row.

Each button press scrambles the colored squares in a complicated, but nonrandom way. Prove that there exist two different sequences of 32 button presses that both produce the same configuration, if the puzzle is initially in the state shown above.

4. 65 distinct integers are chosen in the range $1, 2, 3.....\ 2022$. Prove that there must exist four of the chosen integers (call them a,b, c,d) such that a−b+c−d is a multiple of 2022.