CS21201 Discrete Structures

Practice Problems

Propositional Logic

Validity: The validity of a propositional logic formula means that the formula is true under all possible truth values of propositions.

Satisfiability: The satisfiability of a propositional logic formula means that the formula is true under **at least one** truth value assignment.

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1. **[Submit]** As a student of Discrete Structures (DS), consider the following deduction problem to be coded and solved in propositional logic. Use the following propositions specified in the table below:

Proposition	Meaning
S	TRUE, if you study DS. FALSE otherwise
D	TRUE, if you do well in the DS exam. FALSE otherwise
R	TRUE, if you relax in DS classes. FALSE otherwise
E	TRUE, if you enjoy your third semester. FALSE otherwise

   a. Write the following sentences in propositional logic WITHOUT using the implication ($\rightarrow$) operator. You may use any other operator:

      S1: If you study DS, then you do well in the DS exam
      S2: If you relax in DS classes, then you enjoy your third semester
      S3: You either study in DS classes or relax in DS classes, but not both
      S4: If you study DS, then you don't enjoy your third semester
      S5: If you relax in DS classes, then you don't do well in the DS exam
      G: You enjoy your third semester if and only if you do not do well in the DS exam

   b. Deduce G from $S1 \wedge S2 \wedge S3 \wedge S4 \wedge S5$ using Truth Tables.

2. Someone ate Ronaldo's cookies from his seat in the pavilion. There are three suspects: Messi, Mbappe and Haaland. Ofcourse, all three of them deny stealing. When forced to open their mouths, they say:

Messi: "Ronaldo knows Mbappe, but Haaland liked his cookies."
Mbappe: "I don't even know Ronaldo. Besides, I was on the pitch at that time."
Haaland: "I saw both Messi and Mbappe without Ronaldo in the pavilion at that time, one of them must have done it."

Assume that the two innocent men are telling the truth, but the guilty man may or may not be. Who ate the cookies? Use propositional logic.

3. **Universal Gates**: A universal gate is a boolean logic function that can be used to realize all possible boolean logic functions. NAND and NOR are the two universal gates with their logic as defined below:

   $\text{NAND}(P,Q) = \neg(P \wedge Q)$
   $\text{NOR}(P,Q) = \neg(P \vee Q)$

   Realize the following boolean logic functions using only NAND functions, you may not use the negation ($\neg$) operator:
   a. $P \rightarrow Q$
   b. $P \leftrightarrow Q$
   c. $P \oplus Q$

4. Similar to a universal gate, a **functionally complete connective set** contains connectives which can be combined together to generate all possible boolean logic functions. For example, {NAND} is functionally complete.
   a. Show that $\{\oplus\}$ is not functionally complete
   b. Show that $\{\oplus, \wedge\}$ is functionally complete

5. **[Submit] Resolution Rule**: Given atomic statements p, q and r, the resolution rule states:

$$\begin{array}{c} p \vee q \\ \neg p \vee r \\ \hline \therefore\ q \vee r \end{array}$$

   a. Prove the resolution rule
   b. Establish the validity of the following statements using the resolution rule (alongwith the rules of inference and the laws of logic):

      i. $$\begin{array}{l} p \vee (q \wedge r) \\ p \rightarrow s \\ \hline \end{array}$$

$\therefore\ r \lor s$

ii. $\neg p \lor s$
$\neg t \lor (s \land r)$
$\neg q \lor r$
$p \lor q \lor t$
———
$\therefore\ r \lor s$

6. **Resolution Refutation** is a technique to solve propositional logic problems. In logical deduction, we deduce a goal (G) from true propositions (S1, S2, S3, ...). In resolution refutation, we try to prove that $S1 \lor S2 \lor S3 \lor .... \lor \neg G$ is a contradiction, by applying the resolution rule repeatedly:
    a. Show that Logical Deduction and Resolution Refutation are equivalent.
    b. Solve the following propositional logic problem using resolution refutation (**Hint**: consider converting all the statements to conjunctive normal form first)

$(p \rightarrow q) \rightarrow q$
$(p \rightarrow p) \rightarrow r$
$(r \rightarrow s) \rightarrow \neg(s \rightarrow q)$
———
$\therefore\ r$

7. You are exploring a mysterious island with a reputation for its magical artifacts. As you delve deeper into the island's dense jungle, you come across four ancient altars. Each altar is dedicated to a different element: **Fire**, **Water**, **Earth**, and **Air**. According to local legends, each altar possesses a unique mystical power. The island's residents reveal the following:
    a. The guardian of the Fire altar (F) shares, "If you approach the Water altar, you'll gain insight into the mysteries of Fire. Moreover, if the Earth altar guides you, you'll harness the essence of Fire as well."
    b. The guardian of the Water altar (W) confides, "Neither the Earth altar nor the Air altar holds the secrets of Water's power."
    c. The guardian of the Earth altar (E) declares, "Follow the path to the Fire altar, and you'll be imbued with the energy of the Earth. Opt for the Air altar, and you'll be left empty-handed."
    d. The guardian of the Air altar (A) reveals, "If you seek the strength of Air, the Earth altar will guide you astray. Choose the Fire altar, and you shall forfeit your chance."

The only other information you are aware of is that all these guardians are lying. Can you determine which altar holds the power of **Water**? Use propositional logic to arrive at a conclusion.

8. Prove that $((p \rightarrow q) \land (r \rightarrow \neg q)) \rightarrow (r \rightarrow \neg p)$ is a tautology. Is $((p \rightarrow q) \land (r \rightarrow \neg q))$ logically equivalent to $(r \rightarrow \neg p)$? Prove or disprove.

9. Comment on the **validity** and the **satisfiability** of the following sentences:
    a. (Smoke $\rightarrow$ Fire) $\rightarrow$ ($\neg$Smoke $\rightarrow$ $\neg$Fire)
    b. ((Smoke $\land$ Heat) $\rightarrow$ Fire) $\leftrightarrow$ ((Smoke $\rightarrow$ Fire) $\lor$ (Heat $\rightarrow$ Fire))
    c. Big $\lor$ Dumb $\lor$ (Big $\rightarrow$ Dumb)
    d. (Big $\land$ Dumb) $\lor$ $\neg$ Dumb
    e. (Smoke $\rightarrow$ Fire) $\rightarrow$ ((Smoke $\land$ Heat) $\rightarrow$ Fire)