CS21201 Discrete Structures
Practice Problems

Generating Function

1. Consider the sequence a_0, a_1, a_2,... defined recursively as follows.

   $a_0 = 0$,

   $a_1 = 1$,

   $a_2 = 2$,

   $a_n = 2a_{n-2} + a_{n-3} + 2$ for all $n \geq 3$

   a. Derive a closed-form expression for the (ordinary) generating function $A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ of the sequence.

   b. From the closed-form expression of A(x) derived in Part (a), establish that $a_n = F_{n+2} - 1$ for all $n \geq 0$, where F_0, F_1, F_2,... is the Fibonacci sequence.

2. Consider the sequence a_0, a_1, a_2, a_3,... defined as

   $a_0 = a_1 = 2$,

   $a_n = 2a_{n-1} + 3a_n - 2$ for $n \geq 2$.

   `Find the smallest positive integer k such that $a_n \leq k^n + 1$ for all $n \geq 0$

3. In how many ways can 3000 identical envelopes be divided, in packages of 25, among four student groups so that each group gets at least 150, but not more than 1000, of the envelopes?

4. Show that $(1 - 4x)^{-½}$ generates the sequence ${}^{2n}C_n$ $n \in \mathbf{N}$.

5. **[Submit]** Determine the generating function for the sequence a_0, a_1, a_2,... where a_n is the number of partitions of the nonnegative integer n into

   a. even summands

   b. distinct even summands

   c. distinct odd summands.

6. Find the coefficient of x^{15} in each of the following.

   a. $x^3(1 - 2x)^{10}$

b. $(x^3 - 5x) / (1 - x)^3$

c. $(1 + x)^4 / (1 - x)^4$

7. **[Submit]** In how many ways can two dozen identical robots be assigned to four assembly lines with

    a. at least three robots assigned to each line

    b. at least three, but no more than nine, robots assigned to each line

8. Two cases of soft drinks, 24 bottles of one type and 24 of another are distributed among five surveyors who are conducting taste tests. In how many ways can the 48 bottles be distributed so that each surveyor gets

    a. at least two bottles of each type

    b. at least two bottles of one particular type and at least three of the other

9. If a fair die is rolled 12 times, what is the probability that the sum of the rolls is 30?

10. Determine the following

    a. If a computer generates a random composition of 8, what is the probability the composition is a palindrome

    b. Answer the question in part (a) after replacing 8 by n, a fixed positive integer.

11. Ravi has a biased coin where Pr(H) = $\frac{2}{3}$ and Pr(T) = $\frac{1}{3}$. Assuming that each toss, after the first, is independent of any previous outcome, if Ravi tosses the coin until he gets a tail, what is the probability he tosses it an odd number of times?

12. Determine the sequence generated by each of the following generating functions.

    a. $f(x) = (2x - 3)^3$

    b. $f(x) = x^4 / (1 - x)$

    c. $f(x) = x^3 / (1 - x^2)$

    d. $f(x) = 1 / (1 + 3x)$