Number Representation

... and a few concepts on precision

Pallab Dasgupta
Professor,
Dept. of Computer Sc & Engg

Topics to be Discussed

How are numeric data items actually stored in computer memory?

How much space (memory locations) is allocated for each type of data?

- int, float, char, etc.

How are characters and strings stored in memory?

Number System :: The Basics

We are accustomed to using the so-called *decimal number system*.

- Ten digits :: 0,1,2,3,4,5,6,7,8,9
- Every digit position has a weight which is a power of 10.
- Base or radix is 10.

Example:

$234 = 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$

$250.67 = 2 \times 10^2 + 5 \times 10^1 + 0 \times 10^0 + 6 \times 10^{-1} + 7 \times 10^{-2}$

Binary Number System

Two digits:

- 0 and 1.
- Every digit position has a weight which is a power of 2.
- *Base* or *radix* is 2.

Example:

$$110 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$$

$$101.01 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2}$$

Binary-to-Decimal Conversion

Each digit position of a binary number has a weight.

- Some power of 2.

A binary number:

$B = b_{n-1}\ b_{n-2}\ \ldots.. b_1\ b_0\ .\ b_{-1}\ b_{-2}\ \ldots..\ b_{-m}$

Corresponding value in decimal:

$$D = \sum_{i=-m}^{n-1} b_i 2^i$$

Examples

1. $101011 \rightarrow 1x2^5 + 0x2^4 + 1x2^3 + 0x2^2 + 1x2^1 + 1x2^0$

   $= 43$

   $(101011)_2 = (43)_{10}$

2. $.0101 \rightarrow 0x2^{-1} + 1x2^{-2} + 0x2^{-3} + 1x2^{-4}$

   $= .3125$

   $(.0101)_2 = (.3125)_{10}$

3. $101.11 \rightarrow 1x2^2 + 0x2^1 + 1x2^0 + 1x2^{-1} + 1x2^{-2}$

   5.75

   $(101.11)_2 = (5.75)_{10}$

Decimal-to-Binary Conversion

Consider the integer and fractional parts separately.

For the integer part,

- Repeatedly divide the given number by 2, and go on accumulating the remainders, until the number becomes zero.
- Arrange the remainders *in reverse order*.

For the fractional part,

- Repeatedly multiply the given fraction by 2.
  - Accumulate the integer part (0 or 1).
  - If the integer part is 1, chop it off.
- Arrange the integer parts *in the order* they are obtained.

Example 1 :: 239

2	239	
2	119	--- 1
2	59	--- 1
2	29	--- 1
2	14	--- 1
2	7	--- 0
2	3	--- 1
2	1	--- 1
2	0	--- 1

$$(239)_{10} = (11101111)_2$$

Example 2 :: 64

2	64	
2	32	--- 0
2	16	--- 0
2	8	--- 0
2	4	--- 0
2	2	--- 0
2	1	--- 0
2	0	--- 1

$$(64)_{10} = (1000000)_2$$

Example 3 :: .634

.634 x 2 = 1.268

.268 x 2 = 0.536

.536 x 2 = 1.072

.072 x 2 = 0.144

.144 x 2 = 0.288

:

:

$(.634)_{10} = (.10100......)_2$

Example 4 :: 37.0625

$(37)_{10} = (100101)_2$

$(.0625)_{10} = (.0001)_2$

$\therefore (37.0625)_{10} = (100101\,.\,0001)_2$

Hexadecimal Number System

A compact way of representing binary numbers.

16 different symbols (radix = 16).

0	→	0000	8	→	1000
1	→	0001	9	→	1001
2	→	0010	A	→	1010
3	→	0011	B	→	1011
4	→	0100	C	→	1100
5	→	0101	D	→	1101
6	→	0110	E	→	1110
7	→	0111	F	→	1111

Binary-to-Hexadecimal Conversion

For the integer part,

- **Scan the binary number from *right to left*.**
- **Translate each group of four bits into the corresponding hexadecimal digit.**
    - **Add *leading* zeros if necessary.**

For the fractional part,

- **Scan the binary number from *left to right*.**
- **Translate each group of four bits into the corresponding hexadecimal digit.**
    - **Add *trailing* zeros if necessary.**

Example

1. $(\underline{1011}\ \underline{0100}\ \underline{0011})_2 = (B43)_{16}$

2. $(\underline{10}\ \underline{1010}\ \underline{0001})_2 = (2A1)_{16}$

3. $(.\underline{1000}\ \underline{010})_2 = (.84)_{16}$

4. $(\underline{101}\ .\ \underline{0101}\ \underline{111})_2 = (5.5E)_{16}$

Hexadecimal-to-Binary Conversion

Translate every hexadecimal digit into its 4-bit binary equivalent.

Examples:

$(3A5)_{16} = (0011\ 1010\ 0101)_2$

$(12.3D)_{16} = (0001\ 0010\ .\ 0011\ 1101)_2$

$(1.8)_{16} = (0001\ .\ 1000)_2$

Unsigned Binary Numbers

An n-bit binary number

$B = b_{n-1}b_{n-2} \ldots. b_2b_1b_0$

- 2^n distinct combinations are possible, 0 to 2^n-1.

For example, for n = 3, there are 8 distinct combinations.

- 000, 001, 010, 011, 100, 101, 110, 111

Range of numbers that can be represented

n=8 ➔ 0 to 2^8-1 (255)

n=16 ➔ 0 to 2^{16}-1 (65535)

n=32 ➔ 0 to 2^{32}-1 (4294967295)

Signed Integer Representation

Many of the numerical data items that are used in a program are signed (positive or negative).

- Question:: How to represent sign?

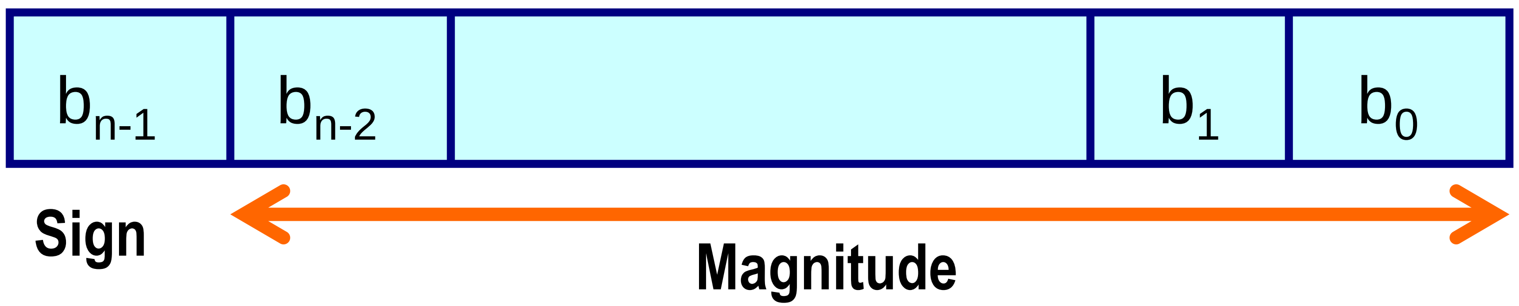
Three possible approaches:

- Sign-magnitude representation
- One's complement representation
- Two's complement representation

Sign-magnitude Representation

For an n-bit number representation

- **The most significant bit (MSB) indicates sign**
  - **0 → positive**
  - **1 → negative**
- **The remaining n-1 bits represent magnitude.**



Contd.

Range of numbers that can be represented:

Maximum :: $+(2^{n-1} - 1)$

Minimum :: $-(2^{n-1} - 1)$

A problem:

Two different representations of zero.

$+0 \rightarrow$ 0 000….0

$-0 \rightarrow$ 1 000….0

One's Complement Representation

Basic idea:

- Positive numbers are represented exactly as in sign-magnitude form.
- Negative numbers are represented in 1's complement form.

How to compute the 1's complement of a number?

- Complement every bit of the number (1→0 and 0→1).
- MSB will indicate the sign of the number.

  0 → positive

  1 → negative

Example :: n=4

0000	→	+0	1000	→	-7
0001	→	+1	1001	→	-6
0010	→	+2	1010	→	-5
0011	→	+3	1011	→	-4
0100	→	+4	1100	→	-3
0101	→	+5	1101	→	-2
0110	→	+6	1110	→	-1
0111	→	+7	1111	→	-0

To find the representation of, say, -4, first note that

+4 = 0100

-4 = 1's complement of 0100 = 1011

Contd.

Range of numbers that can be represented:

Maximum :: $+ (2^{n-1} - 1)$

Minimum :: $- (2^{n-1} - 1)$

A problem:

Two different representations of zero.

+0 → 0 000….0

-0 → 1 111….1

Advantage of 1's complement representation

- Subtraction can be done using addition.
- Leads to substantial saving in circuitry.

Subtraction Using Addition :: 1's Complement

How to compute A – B ?

- **Compute the 1's complement of B (say, B_1).**
- **Compute $R = A + B_1$**
- **If the carry obtained after addition is '1'**
    - **Add the carry back to R (called *end-around carry*).**
    - **That is, $R = R + 1$.**
    - **The result is a positive number.**

Else

- **The result is negative, and is in 1's complement form.**

Example 1 :: 6 – 2

1's complement of 2 = 1101

6	::	0110	A
-2	::	1101	B_1
		1 0011	R
		1	
		0100 → +4	End-around carry

Assume 4-bit representations.

Since there is a carry, it is added back to the result.

The result is positive.

Example 2 :: 3 – 5

1's complement of 5 = 1010

3	::	0011	A
-5	::	1010	B_1
		1101	R

⇩

-2

Assume 4-bit representations.

Since there is no carry, the result is negative.

1101 is the 1's complement of 0010, that is, it represents –2.

Arithmetic Operations: 1's Complement

1's complement of X = $2^n - 1 - X$

Arithmetic	1's complement
$x + y$	$x + y$
$x - y$	$x + (2^n - 1 - y) = 2^n - 1 + (x - y)$
$-x + y$	$(2^n - 1 - x) + y = 2^n - 1 + (-x + y)$
$-x - y$	$(2^n - 1 - x) + (2^n - 1 - y) = 2^n - 1 + (2^n - 1 - x - y)$

1's Complement Example

Example: – 4 – 3 = – 7

4 in binary = 0100.
Flipping the bits, you get –4 (1011) in binary.
3 in binary = 0011.
Flipping the bits, you get –3 (1100) in binary.

  1011 (11 in decimal, or 15-4)
+ 1100 (12 in decimal, or 15-3)

 1,0111 (23 in decimal (15+15-7))
So now take the extra 1 and remove it from the 5th spot and add it to the remainder
   0111
+   1

   1000 (-7 in 1's comp)

Arithmetic Operations: Example: 4 – 3 = 1

$4_{10} = 0100_2$

$3_{10} = 0011_2$ $-3_{10} \rightarrow 1100_2$ in one's complement

0100 (4 in decimal)
+ 1100 (12 in decimal or 15-3)
1,0000 (16 in decimal or 15+1)
0001(after deleting 2^n-1)

We discard the extra 1 at the left which is 2^n and add one at the first bit.

Arithmetic Operations: Example: -4 +3 = -1

$4_{10} = 0100_2$ -4_{10} → Using one's comp. → 1011_2 (Invert bits)

$3_{10} = 0011_2$

1011 (11 in decimal or 15-4)
+ 0011 (3 in decimal)
1110 (14 in decimal or 15-1)

If the left-most bit is 1, it means that we have a negative number.

Overflow: Example: 5 + 6

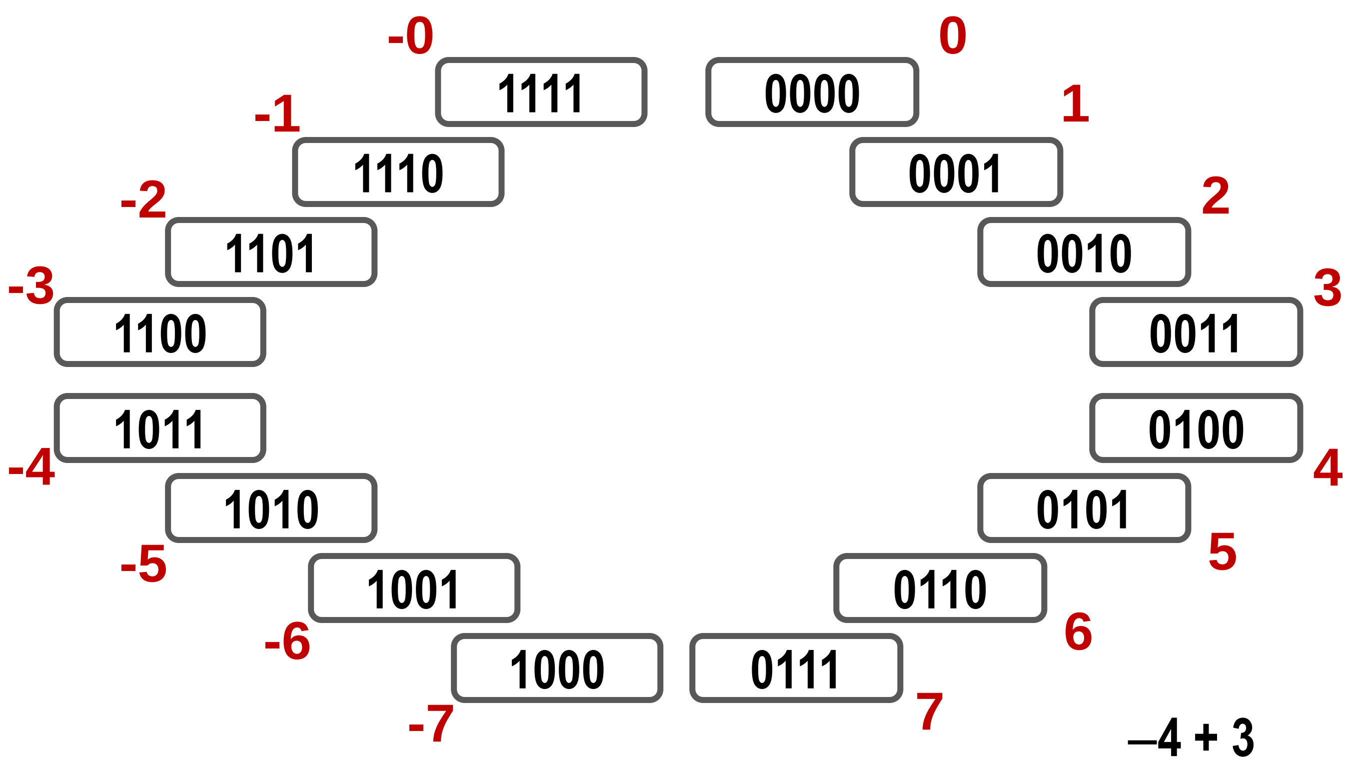
$5_{10} = 0101_2$

$6_{10} = 0110_2$

```
  0101 (5 in decimal )
+ 0110 (6 in decimal )
  1011 (negative numbers in 1's compliment )
```

Overflows are handled separately

Explanation: – 4 + 3



In this example:

1's complement of X = 15 – X

–4 + 3

Start from –4 and take 3 steps clockwise

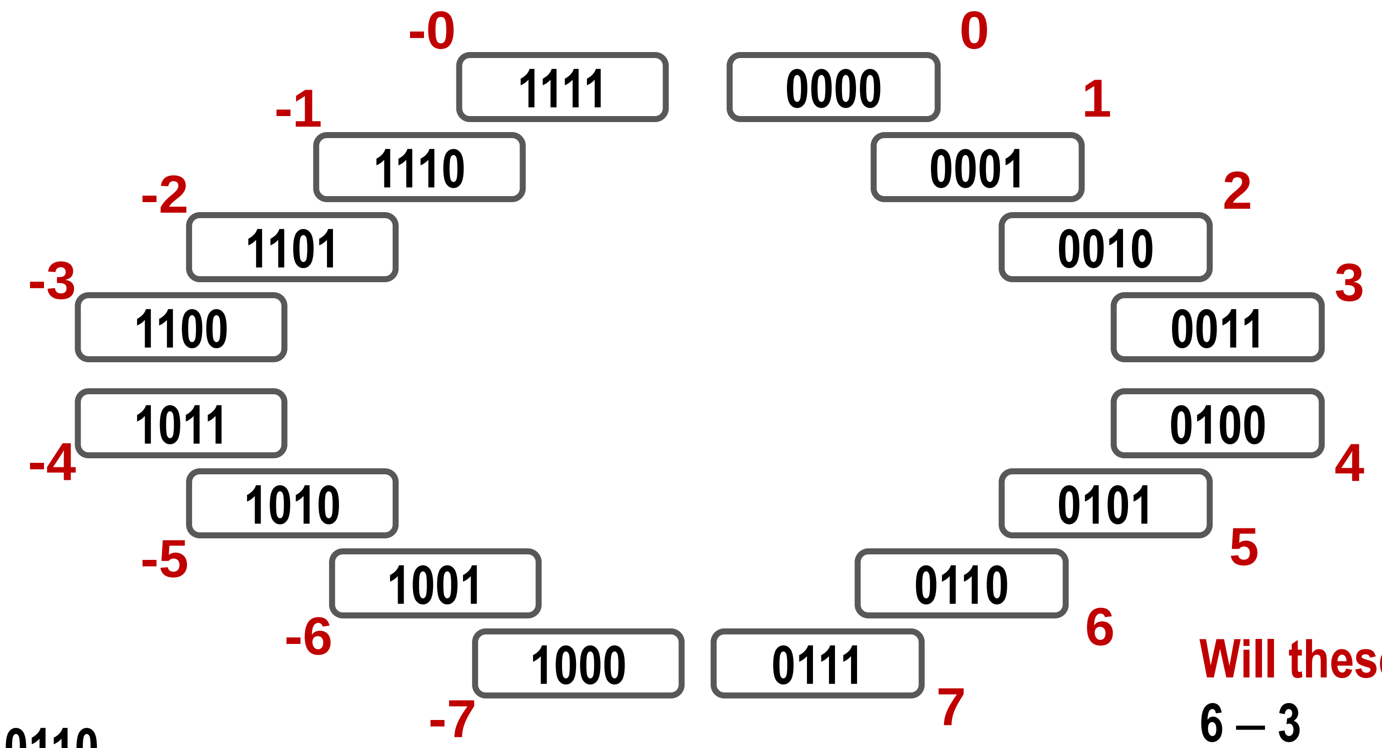
OR

Start from 3 and take (15 – 4) = 11 steps clockwise

Explanation: 6 – 3

In this example:

1’s complement of X = 15 – X




```
  0110
+ 1100
 10010
```

Will these work?

6 – 3

Start from – 3 and take 6 steps clockwise

OR

Start from 6 and take (15 – 3) = 12 steps clockwise

We have end around carry only when we traverse the two 0’s clockwise !!

Adding the end-around carry compensates for the two 0’s !!

How to handle the problem of having two 0's ?

Two's Complement Representation

Basic idea:

- **Positive numbers are represented exactly as in sign-magnitude form.**
- **Negative numbers are represented in 2's complement form.**

How to compute the 2's complement of a number?

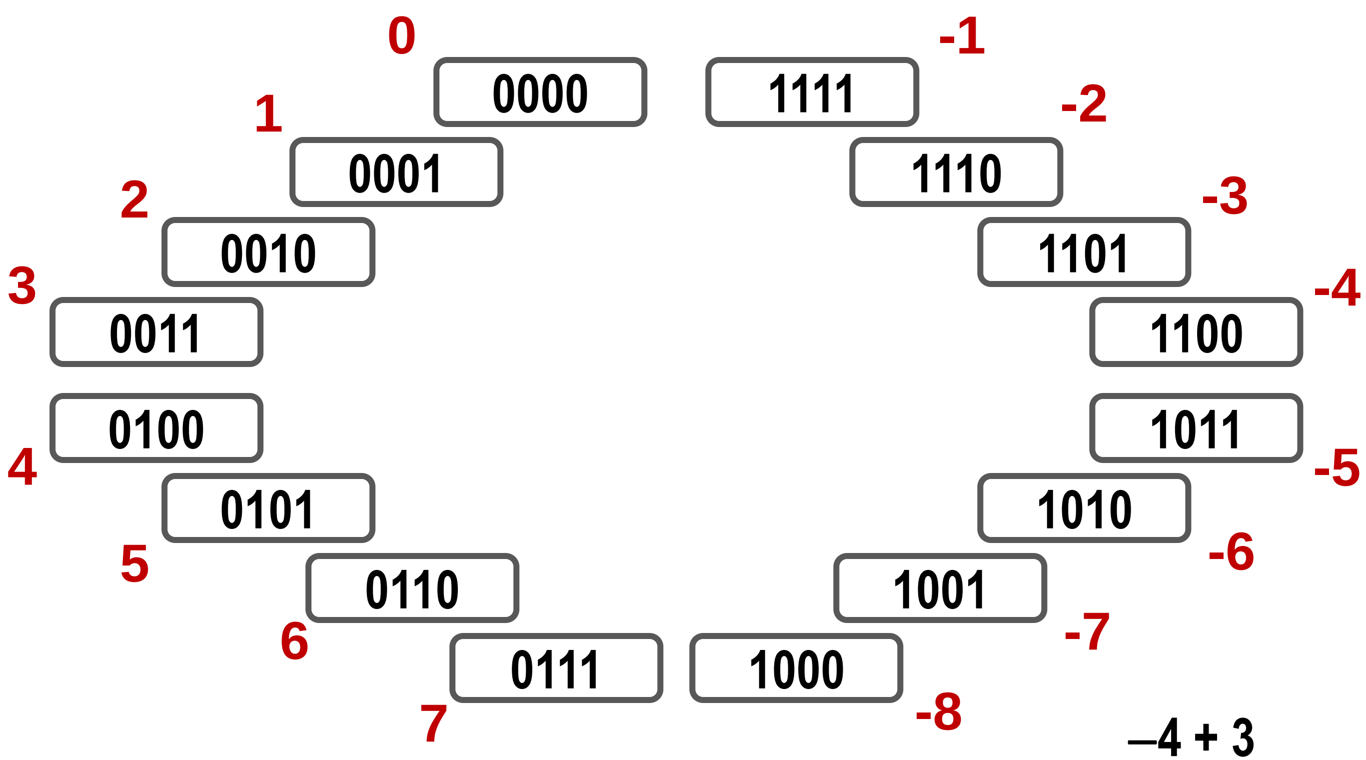
- **Complement every bit of the number (1→0 and 0→1), and then *add one* to the resulting number.**
- **MSB will indicate the sign of the number.**

  0 → positive

  1 → negative

Two's complement: – 4 + 3

2's complement of X in a k-bit representation = $2^k – X$



–4 + 3

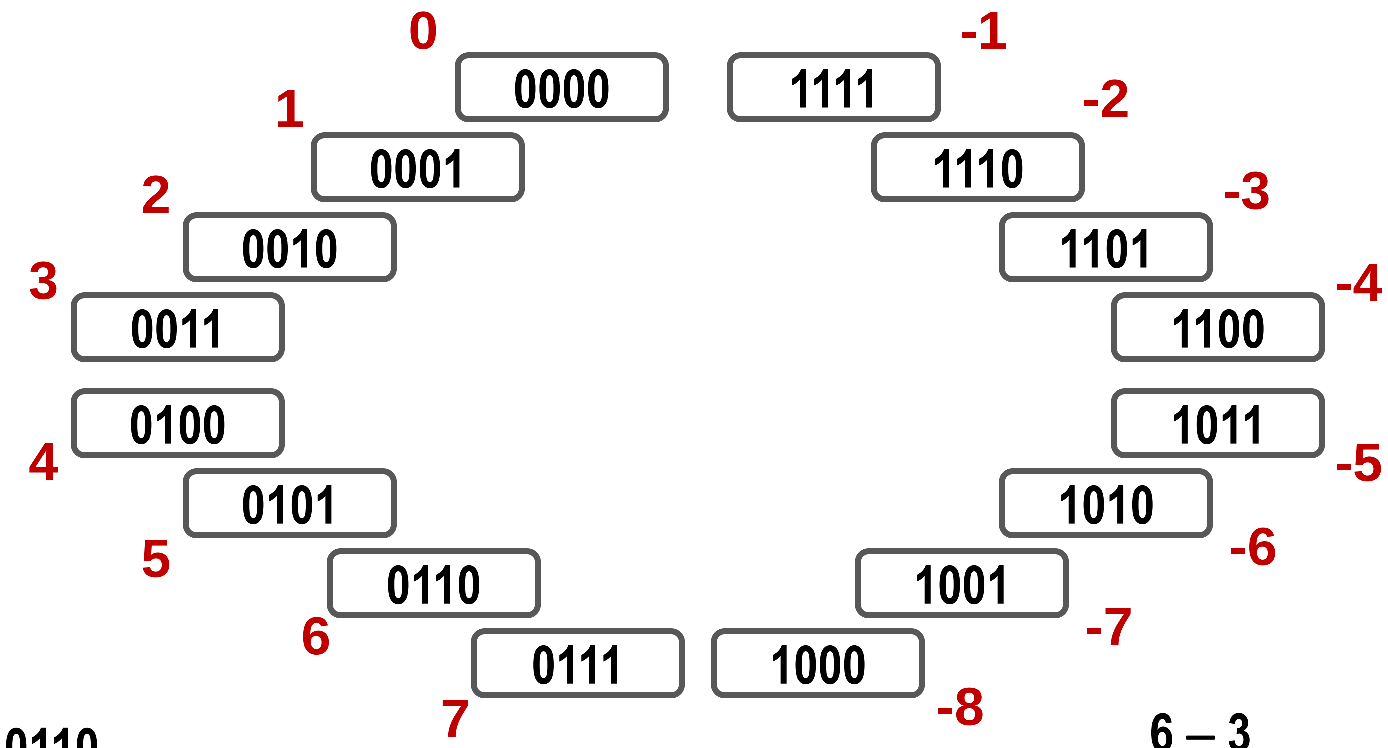
Start from –4 and take 3 steps clockwise

OR

Start from 3 and take (16 – 4) = 12 steps clockwise

Two’s complement: 6 – 3

2’s complement of X in a k-bit representation = $2^k – X$



$$
\begin{array}{r}
0110 \\
+\ 1101 \\
\hline
10011
\end{array}
$$

6 – 3
Start from –3 and take 6 steps clockwise
OR
Start from 6 and take (16 – 3) = 13 steps clockwise

The end around carry can be ignored (modulo arithmetic) !!

Two's complement

Range of numbers that can be represented:

Maximum :: $+(2^{n-1} - 1)$

Minimum :: -2^{n-1}

Advantage:

- ***Unique representation of zero*.**
- **Subtraction can be done using addition.**
- **Leads to substantial saving in circuitry.**

Almost all computers today use the 2's complement representation for storing negative numbers.

Contd.

In C

- short int
  - 16 bits ➔ $+(2^{15}-1)$ to -2^{15}
- int
  - 32 bits ➔ $+(2^{31}-1)$ to -2^{31}
- long int
  - 64 bits ➔ $+(2^{63}-1)$ to -2^{63}

Floating-point Numbers

The representations discussed so far applies only to integers.

- Cannot represent numbers with fractional parts.

We can assume a decimal point before a 2’s complement number.

- In that case, pure fractions (without integer parts) can be represented.

We can also assume the decimal point somewhere in between.

- This lacks flexibility.
- Very large and very small numbers cannot be represented.

Representation of Floating-Point Numbers

A floating-point number F is represented by a doublet <M,E> :

$F = M \times B^E$

- B → exponent base (usually 2)
- M → mantissa
- E → exponent

- M is usually represented in 2's complement form, with an implied decimal point before it.

For example,

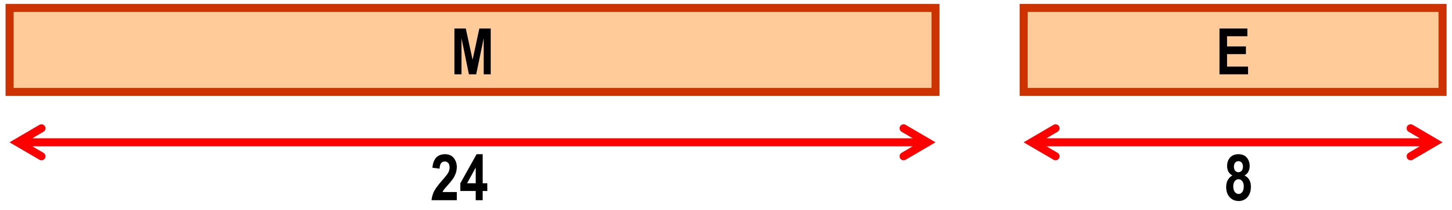
In decimal,

0.235×10^6

In binary,

0.101011×2^{0110}

Example :: 32-bit representation



- M represents a 2's complement fraction

  $1 > M > -1$

- E represents the exponent (in 2's complement form)

  $127 > E > -128$

Points to note:

- The number of *significant digits* depends on the number of bits in M.
  - 6 significant digits for 24-bit mantissa.
- The *range* of the number depends on the number of bits in E.
  - 10^{38} to 10^{-38} for 8-bit exponent.

A Warning

The representation for floating-point numbers as shown is just for illustration.

The actual representation is a little more complex.

In C:

- float :: 32-bit representation
- double :: 64-bit representation

Representation of Characters

Many applications have to deal with non-numerical data.

- Characters and strings.
- There must be a standard mechanism to represent alphanumeric and other characters in memory.

Three standards in use:

- Extended Binary Coded Decimal Interchange Code (EBCDIC)
  - Used in older IBM machines.
- American Standard Code for Information Interchange (ASCII)
  - Most widely used today.
- UNICODE
  - Used to represent all international characters.
  - Used by Java.

ASCII Code

Each individual character is numerically encoded into a unique 7-bit binary code.

- A total of 2^7 or 128 different characters.
- A character is normally encoded in a byte (8 bits), with the MSB not been used.

The binary encoding of the characters follow a regular ordering.

- Digits are ordered consecutively in their proper numerical sequence (0 to 9).
- Letters (uppercase and lowercase) are arranged consecutively in their proper alphabetic order.

Some Common ASCII Codes

‘A’ :: 41 (H) 65 (D)

‘B’ :: 42 (H) 66 (D)

………..

‘Z’ :: 5A (H) 90 (D)

‘a’ :: 61 (H) 97 (D)

‘b’ :: 62 (H) 98 (D)

………..

‘z’ :: 7A (H) 122 (D)

‘0’ :: 30 (H) 48 (D)

‘1’ :: 31 (H) 49 (D)

………..

‘9’ :: 39 (H) 57 (D)

‘(‘ :: 28 (H) 40 (D)

‘+’ :: 2B (H) 43 (D)

‘?’ :: 3F (H) 63 (D)

‘\n’ :: 0A (H) 10 (D)

‘\0’ :: 00 (H) 00 (D)

String Representation in C

In C, the second approach is used.

- The ‘\0’ character is used as the string delimiter.

Example:

“Hello” ➔

H	e	l	l	o	‘\0’

A null string “” occupies one byte in memory.

- Only the ‘\0’ character.

Precision of Floating Point Numbers

- Two floating point numbers should not be checked for equality

```
double x = 10;
double y = sqrt(x);
y *= y;
if (x == y)
      printf("Square root is exact \n");
else
      printf("%lf\n", x - y);

Output:  -1.778636e-015
```

- Use a *tolerance* to compare equality. Here is some threshold that defines what is "close enough" for equality

```
double tolerance = 0.0000001f
if (fabs(x - y) < tolerance) {...}
```

Precision of Floating Point Numbers (contd..)

- Subtraction of two floating point numbers create a problem when they are *nearly equal*

```
for (i = 1; i < 20; ++i)
{
      double h = pow(10.0, -i);
      printf("%lf \n",(sin(1.0+h) - sin(1.0))/h));
}
printf("True result: %lf", cos(1.0));
```

Output

```
0.4..........
0.53.........
0.53.........
0.5402.......
0.5402.......
0.540301.....
0.5403022....
0.540302302...
0.54030235....
0.5403022.....
0.540301......
0.54034.......
0.53..........
0.544.........
0.55..........
0
0
0
0
True result:
0.54030230586814
```

Observation

- The precision of the calculated value improves initially
- The precision decreases gradually when values of *sin(1.0 + h)* and *sin(1.0)* become nearly equal to each other

Precision of Floating Point Numbers (contd..)

- **Floating point numbers have finite ranges**

```
float val = 16777216;
printf("val = %f \n (val + 1) = %f", val, (val + 1));
```

```
Output:  val = 16777216
        (val + 1) = 16777216
```

Observation

- In both the cases, same number is printed
- In 32 bit floating point representation, 24 bits are used to represent the mantissa
- In this case, *16777216* = *2^24.* The 32 bit representation of the floating point does not have any precision left to represent 16777216 + 1