Graph Coloring

K-coloring

- A *k-coloring* of G is a labeling $f{:}V(G) \rightarrow \{1,\ldots,k\}$.
  - The labels are *colors*
  - The vertices with color *i* are a *color class*
  - A *k*-coloring is *proper* if $x \leftrightarrow y$ implies $f(x) \neq f(y)$
  - A graph G is *k-colorable* if it has a proper *k-coloring*
  - The *chromatic number* $\chi(G)$ is the maximum *k* such that G is *k*-colorable
  - If $\chi(G) = k$, then G is *k-chromatic*
  - If $\chi(G) = k$, but $\chi(H) < k$ for every proper subgraph *H* of *G*, then *G* is *color-critical* or *k-critical*

Order of the largest clique

- Let $\alpha(G)$ denote the *independence number* of G, and $\omega(G)$ denote the order of the largest complete subgraph of G.
  - $\chi(G)$ may exceed $\omega(G)$. Consider $G = C_{2r+1} \vee K_s$

Cartesian Product

- The *Cartesian product* of graphs G and H, written as G□H, is the graph with vertex set V(G) X V(H) specified by putting (u,v) adjacent to (u',v') if and only if
  - (1) $u = u'$ and $vv' \in E(H)$, or
  - (2) $v = v'$ and $uu' \in E(G)$

A graph G is *m*-colorable if and only if $G \square K_m$ has an idependent set of size n(G).

Also: $\chi(G \square H) = \max\{ \chi(G), \chi(H) \}$

Algorithm Greedy-Coloring

- The greedy coloring with respect to a vertex ordering $v_1, \ldots, v_n$ of $V(G)$ is obtained by coloring vertices in the order $v_1, \ldots, v_n$, assigning to v_i the smallest indexed color not already used on its lower-indexed neighbors.

Results

- $\chi(G) \leq \Delta(G) + 1$
- If G is an interval graph, then $\chi(G) = \omega(G)$
- If a graph G has degree sequence $d_1 \geq \ldots \geq d_n$, then $\chi(G) \leq 1 + \max_i \min\{ d_i, i-1\}$

More results

- If H is a *k*-critical graph, then $\delta(H) \geq k-1$
- If G is a graph, then $\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H)$
- Brooks Theorem:

  If G is a connected graph other than a clique or an odd cycle, then $\chi(G) \leq \Delta(G)$.

Mycielski's Construction

- Mycielski found a construction that builds from any given *k*-chromatic triangle-free graph G a *k+1*-chromatic triangle-free supergraph G′.
  - Given G with vertex set $V = \{v_1,\ldots,v_n\}$, add vertices $U = \{u_1,\ldots,u_n\}$ and one more vertex w.
  - Beginning with $G'[V] = G$, add edges to make u_i adjacent to all of $N_G(v_i)$, and then make $N(w) = U$. Note that U is an independent set in G′.

Critical Graphs

- Suppose that G is a graph with $\chi(G) > k$ and that X,Y is a partition of V(G). If G[X] and G[Y] are *k*-colorable, then the edge cut [X,Y] has at least *k* edges.

- [Dirac] Every *k*-critical graph is *k*–1 edge-connected.

Critical Graphs

Suppose S is a set of vertices in a graph G. An *S*-component of G is an induced subgraph of G whose vertex set consists of *S* and the vertices of a component of G – S.

- If G is *k*-critical, then G has no cutset of vertices inducing a clique. In particular, if G has a cutset S={x,y}, then x and y are not adjacent and G has an *S*-component H such that $\chi(H + xy) \geq k$.

Chromatic Recurrence

The function $\chi(G;k)$ counts the mappings f: $V(G) \rightarrow [k]$ that properly color G from the set $[k] = \{1,\ldots,k\}$. In this definition, the *k*-colors need not all be used, and permuting the colors used produces a different coloring.

- If G is a simple graph and $e \in E(G)$, then
  $$\chi(G; k) = \chi(G - e; k) - \chi(G.e; k)$$

Line Graphs

The *line graph* of G, written *L(G)*, is a simple graph whose vertices are the edges of G, with $ef \in E(L(G))$ when *e* and *f* share a vertex of G.

- An Eulerian circuit in G yields a spanning cycle in L(G). The converse need not hold
- A matching in G is an independent set in L(G); we have $\alpha'(G) = \alpha(L(G))$

Edge Coloring

A *k-edge-coloring* of G is a labeling *f*: E(G) → [k]

- The labels are *colors*
- The set of edges with one color is a *color class.*
- A *k*-edge-coloring is *proper* if edges sharing a vertex have different colors; equivalently, each color class is a matching
- A graph is *k-edge-colorable* if it has a proper *k*-edge-coloring
- The *edge-chromatic-number* χ'(G) of a loop-less graph G is the least *k* such that G is k-edge-colorable

Results

- $\chi'(G) \geq \Delta(G)$.
- If G is a loop-less graph, then $\chi'(G) \leq 2\Delta(G) - 1$.
- If G is bipartite, then $\chi'(G) = \Delta(G)$.

A regular graph G has a $\Delta(G)$-edge coloring if and only if it decomposes into 1-factors. We say that G is *1-factorable.*

- Every simple graph with maximum degree Δ has a proper Δ+1-edge-coloring.