Non-Regular Languages & Pumping Lemma

CS 510 – Fall 2005

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We would like to show that there are languages that are *not regular* and they satisfy the Pumping Lemma.

Let us consider the language

$$L = \{ab^jc^j \mid j \geq 0\} \cup \{a^ib^jc^k \mid i,j,k \geq 0 \wedge i \neq 1\}$$

Let us show that this language satisfies the pumping lemma. This means that we need to show that there exists a constant n (pumping length) such that, for each string $w \in L$ and $|w| \geq n$ we have that

- there are x, y, z such that $w = xyz$
- $|xy| \leq n$
- $|y| \geq 1$
- for each $i \geq 0$ we have that $xy^iz \in L$

Let us take $n = 2$ and let us show that this pumping length allows us to meet the necesary conditions. Let us consider the possible strings of length greater or equal to 2 that are in L:

- $w = ab^jc^j$ with $j \geq 1$ is a string of proper length; in this case, we can take $x = \epsilon$, $y = a$ and $z = b^jc^j$ and let us show that this partition meets all the necessary conditions
  - $|xy| = |a| = 1 \leq 2$
  - $|y| = |a| = 1 \geq 1$
  - if we take any $i \geq 0$, we have that xy^iz is in L (since it is of the form $a^ib^jc^j$)
- $w = aab^jc^k$; in this case we can take $x = \epsilon$, $y = aa$ and $z = b^jc^k$. This partition meets all the requirements since
  - $|xy| = |aa| = 2 \leq 2$
  - $|y| = |aa| = 2 \geq 1$
  - for each $i \geq 0$ we have that xy^iz is of the form $(aa)^ib^jc^k$ which is a string of L

- $w = a^r b^j c^k$ with $r > 2$, $j, k \geq 0$; in this case we can use the partition $x = \epsilon$, $y = a$, and $z = a^{r-1} b^j c^k$. Let us show that this partition meets the conditions:
  - $|xy| = |a| = 1 \leq 2$
  - $|y| = |a| = 1 \geq 1$
  - for each $i \geq 0$ we have that $xy^i z$ is of the form $a^{i+r-1} b^j c^k$ which is a string of L
- $w = b^j c^k$ with $j \geq 1$ and $k \geq 0$ and $j + k \geq 2$; in this case we can use the partition $x = \epsilon$, $y = b$, and $z = b^{j-1} c^k$. Let us show that this satisfies the conditions
  - $|xy| = |b| = 1 \leq 2$
  - $|y| = |b| = 1 \geq 1$
  - for each $i \geq 0$ we have that $xy^i z$ is the string $b^{j+i-1} c^k$ which is a string of L
- $w = c^k$ with $k \geq 2$; we can take the partition $x = \epsilon$, $y = c$, and $z = c^{k-1}$. This partition meets the requirements:
  - $|xy| = |c| = 1 \leq 2$
  - $|y| = |c| = 1 \geq 1$
  - for each $i \geq 0$, the string $xy^i z$ is equal to c^{i+k-1} which is a string of L

Thus the language satisfies the pumping lemma conditions.

On the other hand, this language is not regular. We can prove this by using the following proof by contraddiction.

Let us assume L to be regular. We know that the language described by each regular expression is also regular. Thus, we know that $L(ab^*c^*)$ is a regular language. In addition, we know that regular languages are closed under the intersection operation. This means that $L \cap L(ab^*c^*)$ is regular. On the other hand,

$$L \cap L(ab^*c^*) = \{ab^j c^j \mid j \geq 0\}$$

Let us now show that this resulting language does not satisfy the pumping lemma. If we take any n, the string $ab^n c^n$ is a string of $L \cap L(ab^*c^*)$. Furthermore, $|ab^n c^n| = 1 + 2n \geq n$, thus this strong is long enough. If we try to partition the string $ab^n c^n = xyz$ with $|xy| \leq n$ and $|y| \geq 1$ we have only two possible partitions:

1. in the first case, $x = ab^r$ and $y = b^s$ with $s \geq 1$; in this case, we consider $i = 0$ and we obtain the string

   $$xz = ab^{n-s} c^n$$

   this string is not in the language $L \cap L(ab^*c^*)$

2. in the second case we have $x = \epsilon$ and $y = ab^j$ $(j \geq 0)$ and $z = b^{n-j}c^n$. If we take $i = 0$ we obtain the string $xz = b^{n-j}c^n$ which is not in the language $L \cap L(ab^*c^*)$

Thus, we cannot find a partition that is going to work for this string, which implies that the language $L \cap L(ab^*c^*)$ is not a regular language, which means we have a contradiction. Since the only assumption we made was that L is regular, this means that such assumption was wrong, i.e., L is not regular.