

1. (a) TRUE

$\nexists R \in \#P$  then  $\exists M$ , a poly-time deterministic TM to decide  $R$ .

$\Rightarrow$  Every  $\#P$  is also an  $NP$ .

$\Rightarrow$  To verify  $w \in R$  we can use  $M$  | Since  $M$  is det. poly time. The Yes answer can be  
To verify  $w \in \bar{R}$  we can also use  $M$  | verified using  $M$  in  $P$ .

(b) TRUE

$\forall w_1 \in L_1$  and  $w_2 \in L_2$ ,  $\exists M_1$  to decide  $w_1 \in L_1$  and  $\exists M_2$  to decide  $w_2 \in L_2$ , where  
 $L_1, L_2 \in TIME$

$\nexists f_1(w_1)$  and  $f_2(w_2)$  are complexities of  $M_1, M_2$  then computing  $w \in L_1 \cap L_2$   
can be achieved as follows:

Run  $w_1$  on  $M_1$  —  $f_1(w_1)$   
Run  $w_2$  on  $M_2$  —  $f_2(w_2)$

$\nexists$  both return a Yes, return a Yes else return a No. —  $O(1)$

Total time  $f_1(w) + f_2(w)$   
 $\downarrow$   $\downarrow$   
Polynomial Polynomial

$\therefore L_1 \cap L_2 \in P$

(c) False.

$SAT \in NP$ -Complete

Graph coloring for interval graphs  $\in P$ .

$GC \leq_P SAT$  For  $k$  colors, for each vertex  $v_i \in G$ ,  $j \in \{1, 2, \dots, k\}$

Create propositions  $c_{ij}$   
Constraints: (a)  $(c_{ij} \wedge \bar{c}_{hj}) \vee (\bar{c}_{ij} \wedge c_{hj}) \vee (\bar{c}_{ij} \wedge \bar{c}_{hj})$  for  $(v_i, v_h) \in E$   
(b) For each vertex a color is assigned, and only one color is given

$B \in \text{NP-Complete}$

$A \in P$

$B \leq_p A$

Hence a polynomial function  $f(w)$  exists transforming an instance  $w \in B$  into  $f(w) \in A$ , solved using a solver for  $A$  in polytime. Hence  $B \in P$

Since  $B \in P$ , now all other problems in  $\text{NP}$  reduce to  $B$  and therefore to  $A$ .

$\therefore \text{NP} = P$ .

(e) TRUE

An  $O(f(n))$  space NTM can be simulated by an  $O(f^2(n))$  space DTM, where  $n$  is the input size

Problems in  $\text{NP}$  take polytime to solve. In polytime  $g(n)$ , space used (cells of the TM) must be  $f(n) \leq g(n)$ , since the TM cannot have visited more cells than  $g(n)$ .

$\therefore \text{NP} \subseteq \text{PSPACE}$

2. (a)  $\text{SAT} \in \text{NP}$

(b)  $\text{VALIDITY} \in \text{CO-NP}$

(c)  $k\text{-CUT} \in P$

(d)  $\text{Non-Validity} \in \text{NP}$

(e)  $k\text{-REGALLOC} \in P$

3. (a)  $\text{SET-COVER} \in \text{NP}$

Certificate is a cover  $C$ .

Compute union of sets in  $C$  (Size of  $C$ )

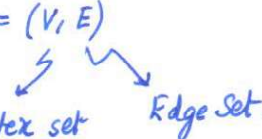
Check in union is the universe (Size of  $U$ )

Polytime. Verifier.

Compute  $C$  Non-Deterministically.

For each  $S_i \in S$ , non-deterministically decide if  $S_i \in C$  or not. | Size of  $S$

Reduction VERTEX-COVER  $\leq_p$  SET-COVER

Consider graph  $G = (V, E)$   
  
 Vertex set      Edge Set.

For each edge  $e_i \in E$  construct  $U$  as follows

$$U = \{e_i \mid e_i \in E\}$$

Construct  $S$  as follows.

For each vertex  $v_j \in V$ ,  $V_j$  is added to  $S$ , where  $V_j$  contains all edges incident on vertex  $v_j \in V$

PART-1 . VERTEX COVER has a solution if the instance of SET COVER has a solution

Consider a  $k$ -sized set cover  $C$ .

Each subset in  $C$  corresponds to a unique vertex in the Graph.  
 i.e.  $V_j$  corresponds to  $v_j$ .

Choose  $v_j$  to be in the vertex cover, if  $V_j \in C$

$$\bigcup_{V \in C} V = U \quad \rightarrow \text{the universal set of all edges.}$$

$\therefore$  By choosing  $v_j$  in the vertex cover all edges are covered.

PART-2 : If VERTEX-COVER has a solution then SET-COVER has a sol<sup>n</sup>.

Consider a  $k$ -sized VC and for each vertex  $v_j$  in the vertex cover pick the subset  $V_j$  and add  $V_j$  to  $C$ . Since all edges are covered by the VC, the union of subsets in  $C$  cover all objects in  $U$ .

But VERTEX-COVER  $\in$  NP-Complete.  $\therefore$  SET-COVER  $\in$  NP-Complete.

(b) The problem is in Difference Polynomial.

$$k\text{-SC} = \{ \langle U, S, k \rangle \mid U \text{ has a cover of size } k \text{ from subsets } \overset{\text{in}}{S} \}$$

$$\overline{k\text{-SC}} = \{ \langle U, S, k \rangle \mid U \text{ has no cover of size } k-1 \text{ from subsets in } S \}$$

$$k\text{-MIN-SC} = k\text{-SC} \cap \overline{k\text{-SC}}$$