In Search of Quantum Advantage: Total Boolean Functions

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65th Birthday Celebration of Prof. S.P. Pal
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Question: How much Computational Advantage does **Quantumness** provide?

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(a) Shor's Factoring Algorithm

(b) Grover's Search

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Only quadratic but unconditional

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QA. Conditional, Debatable

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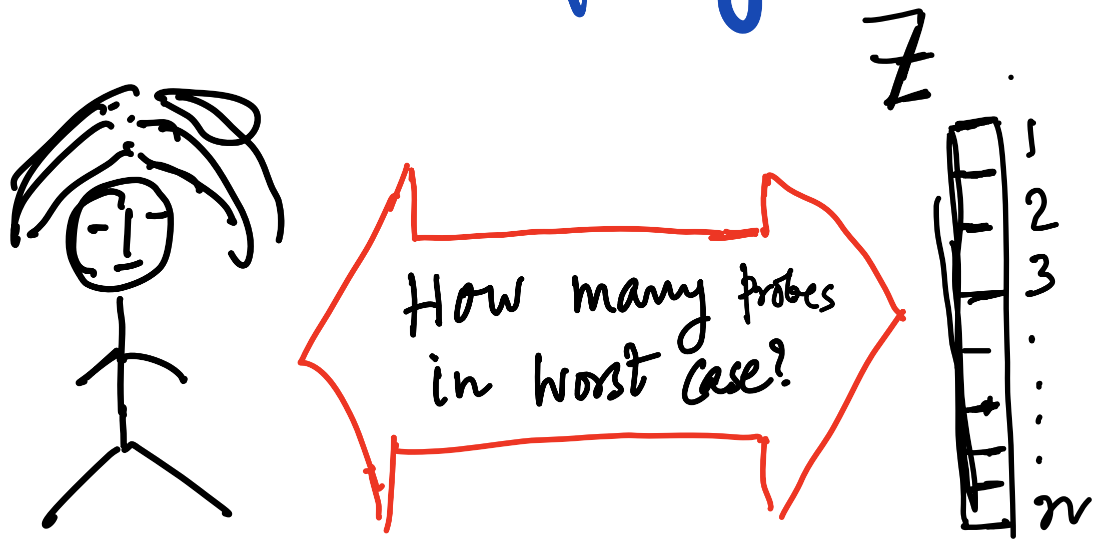
Only quadratic but unconditional

Goal: Unconditional exponential QA, for total functions.

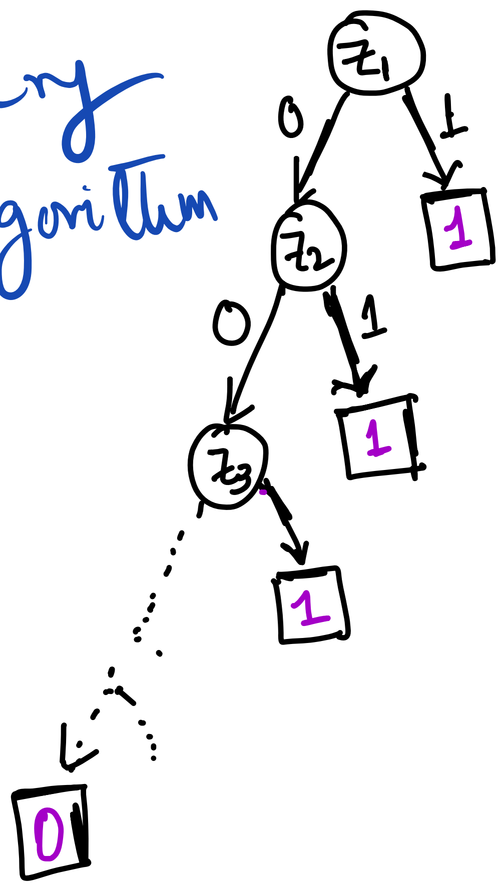
Simple Model - Query

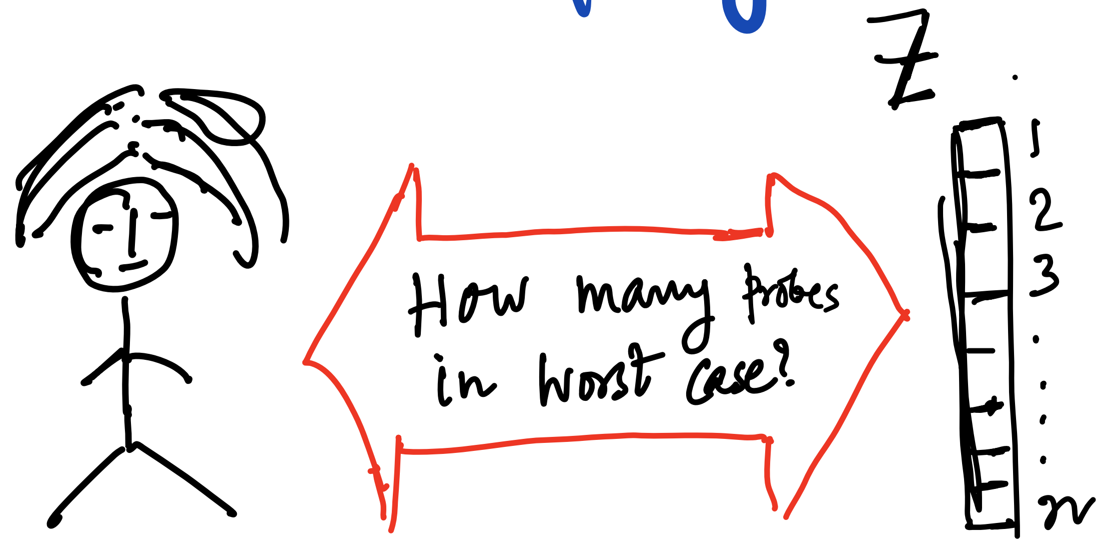


Is there a 1?

Simple Model - Query

Query Algorithm





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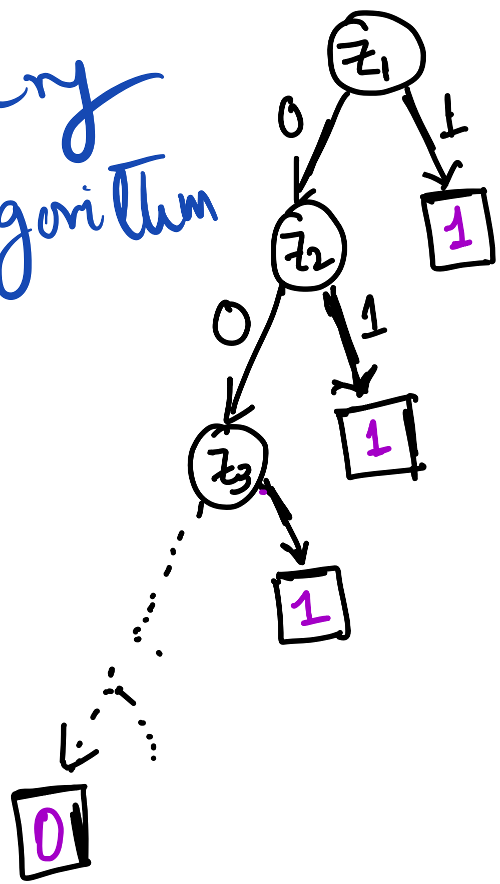

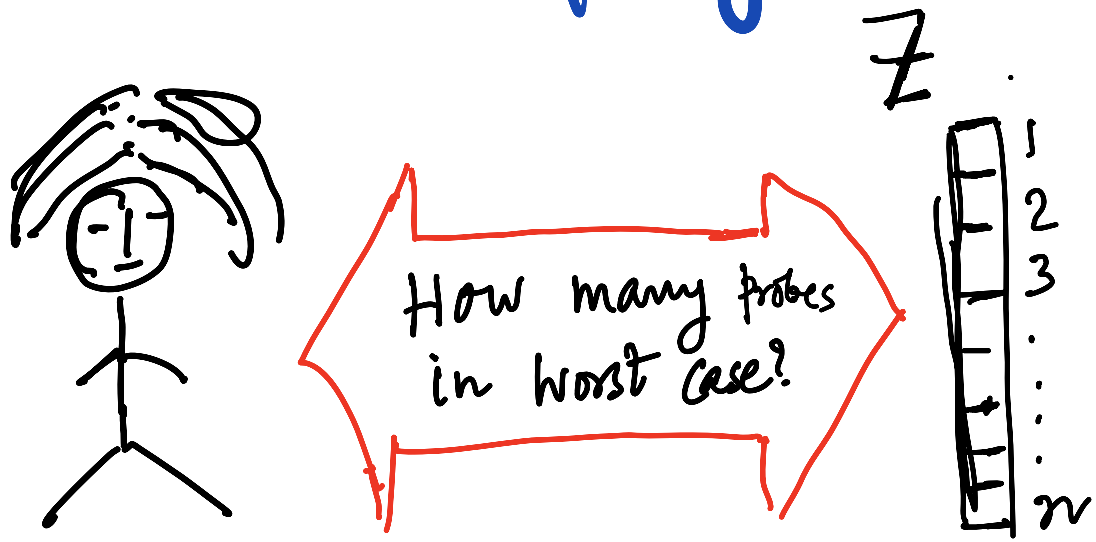

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$D^{dt}(OR) = n$ $\qquad$ $R_{\epsilon}^{dt}(OR) = \Theta(n)$

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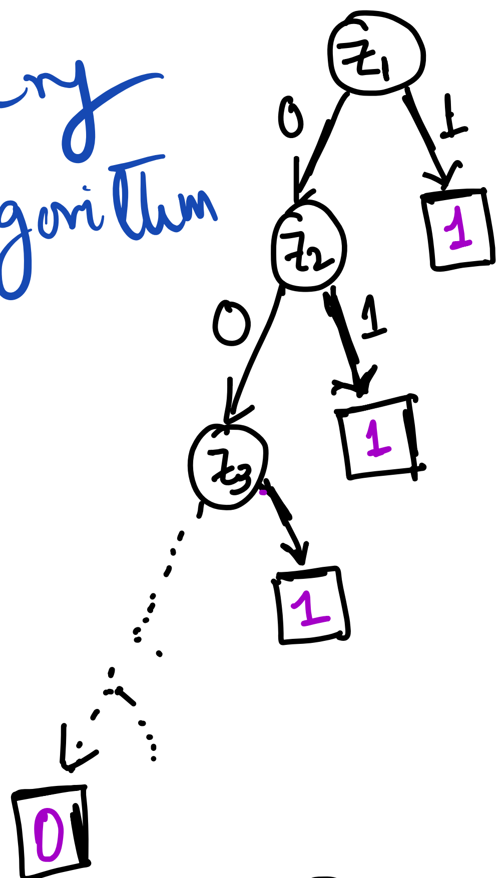

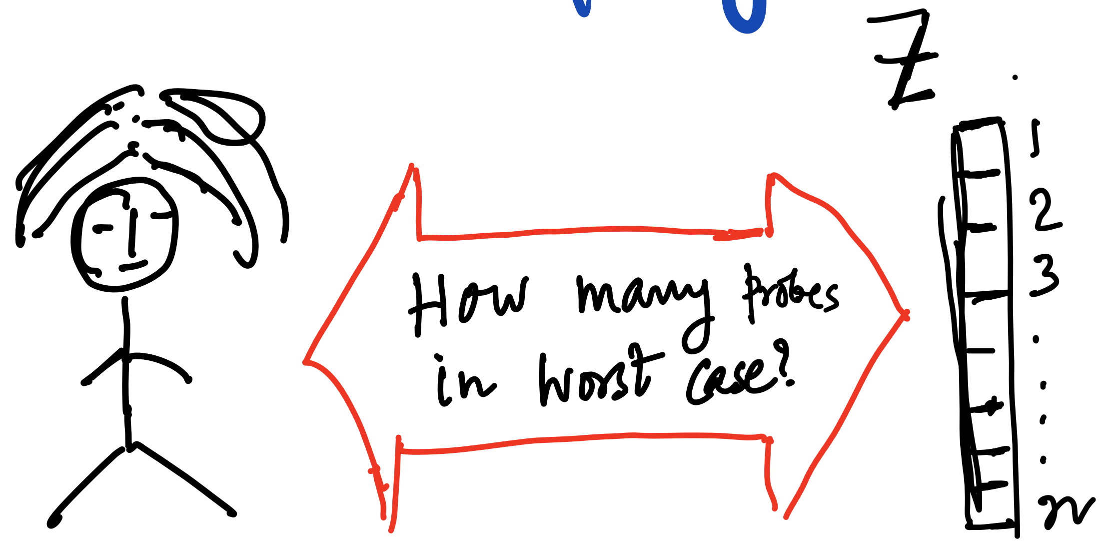

Is there a 1?

$D^{dt}(OR) = n$ $\qquad R_{\epsilon}^{dt}(OR) = \Theta(n)$

Does Quantumness help?

Grover Search: $Q_{\epsilon}^{dt}(OR) = O(\sqrt{n})$ Is this optimal?

Polynomials

T : Deterministic DT computing f.

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$$T(x_1 \cdots x_n) := \sum_{\ell \in \mathcal{L}_T^1} \mathbb{1}_\ell(x_1 \cdots x_n) = f(x_1 \ldots x_n)$$

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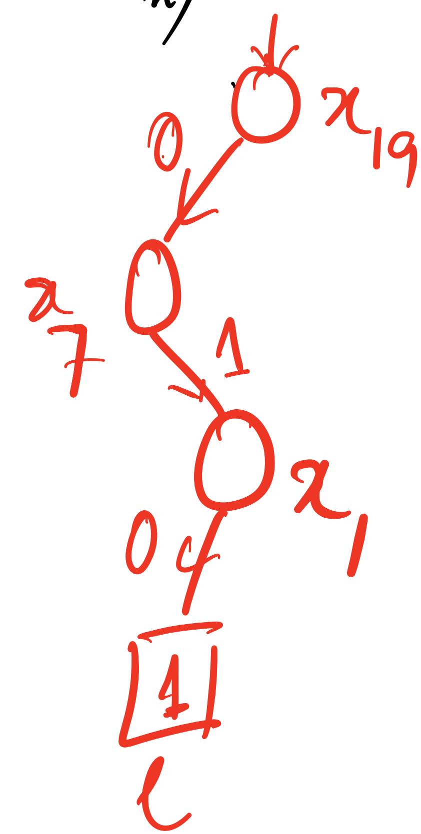

$$\mathbb{1}_\ell(x_1 \cdots x_n) := (1 - x_{19})\, x_7\, (1 - x_1)$$

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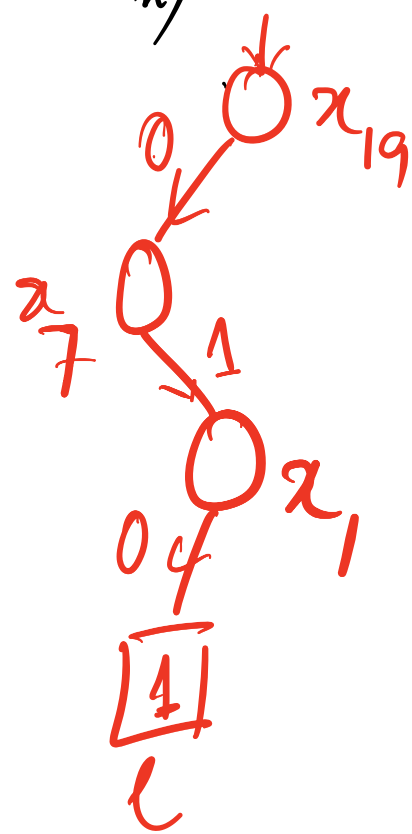

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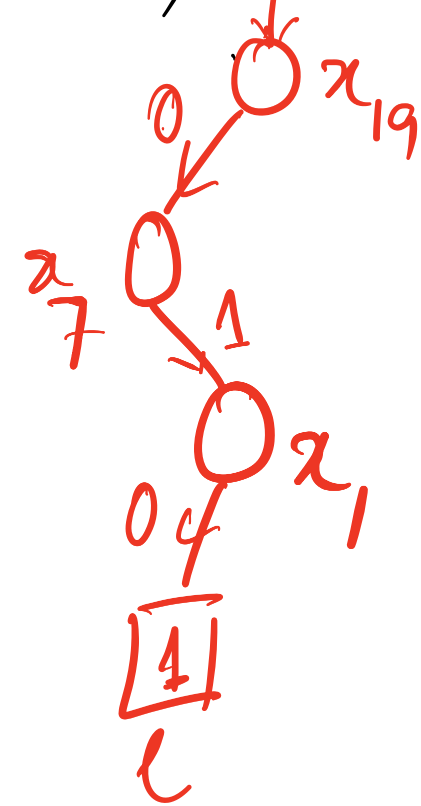

$$\deg(T) \le \mathrm{depth}(T)$$

$$\deg(OR) = n \Rightarrow D^{dt}(OR) = n$$

ϵ-Approximating Polynomials

$\mathcal{T}$: **Randomized** DT computing f; error ϵ.

$$P(x_1 \cdots x_n) := \sum_{T \in \mathcal{T}} \underbrace{\lambda_T}_{\text{Prob } T \text{ is sampled}} T(x_1 \cdots x_n)$$

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Obs: $P(x_1 \cdots x_n)$ = probability $\mathcal{T}$ accepts input $x_1 \cdots x_n$.

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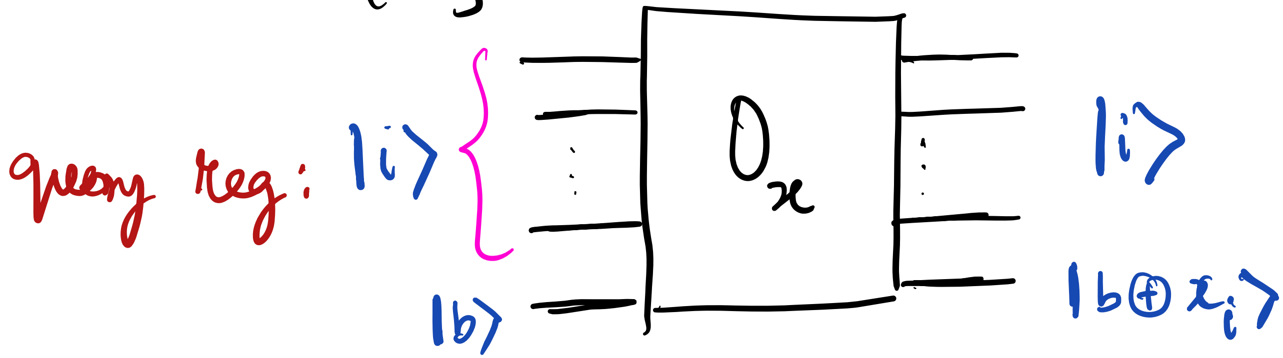
Obs: $P(x_1 \cdots x_n)$ = probability $\mathcal{T}$ accepts input $x_1 \cdots x_n$.

Corollary: $\forall x_1 \cdots x_n : \left| f(x_1 \cdots x_n) - P(x_1 \cdots x_n) \right| \leq \epsilon$

$$\deg(P) \leq \operatorname{depth}(\mathcal{T}) := \max_{T \in \mathcal{T}} \operatorname{depth}(T)$$

Quantum Query Algorithms

$x \in \{0,1\}^n$



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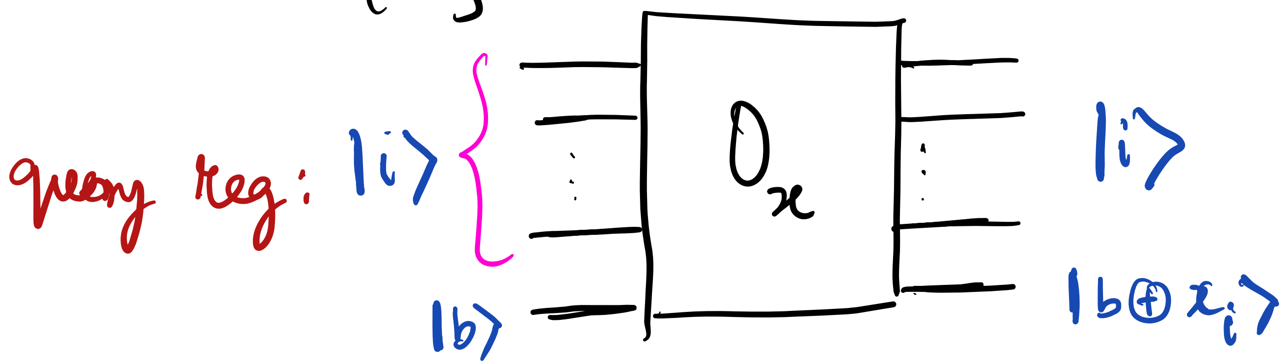

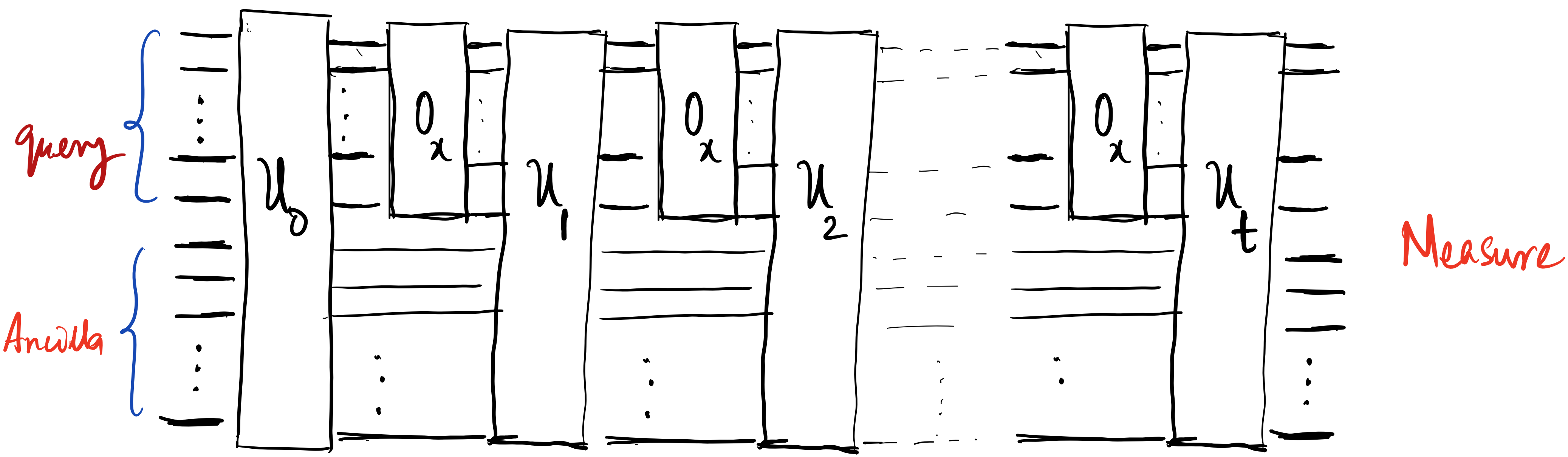

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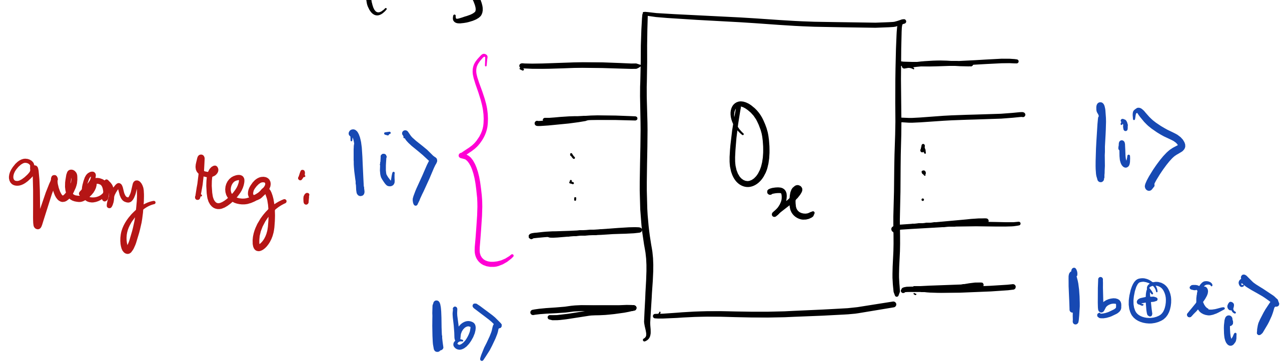

$$\mathcal{U}_t \, O_x \, \mathcal{U}_{t-1} \cdots O_x \, \mathcal{U}_2 \, O_x \, \mathcal{U}_1 \, O_x \, \mathcal{U}_0 \, |0\rangle$$

$|\psi_t\rangle$ $\qquad\qquad\qquad\qquad$ $|\psi_0\rangle$

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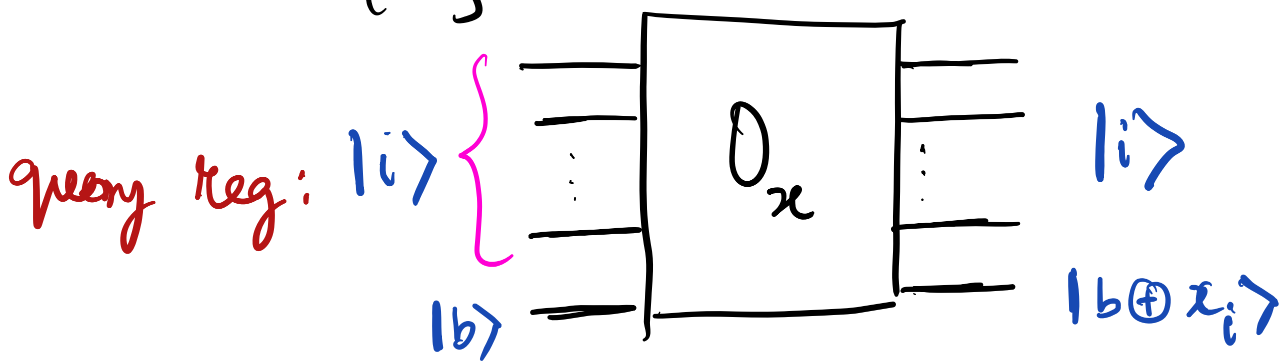

$$\mathcal{U} \quad U_t \, O_x \, U_{t-1} \cdots O_x \, U_2 \, O_x \, U_1 \, O_x \, U_0 \, |0\rangle$$

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Observation: $$|\psi_f\rangle = \sum_{z \in \{0,1\}^{q+a}} p(x) \, |z\rangle$$

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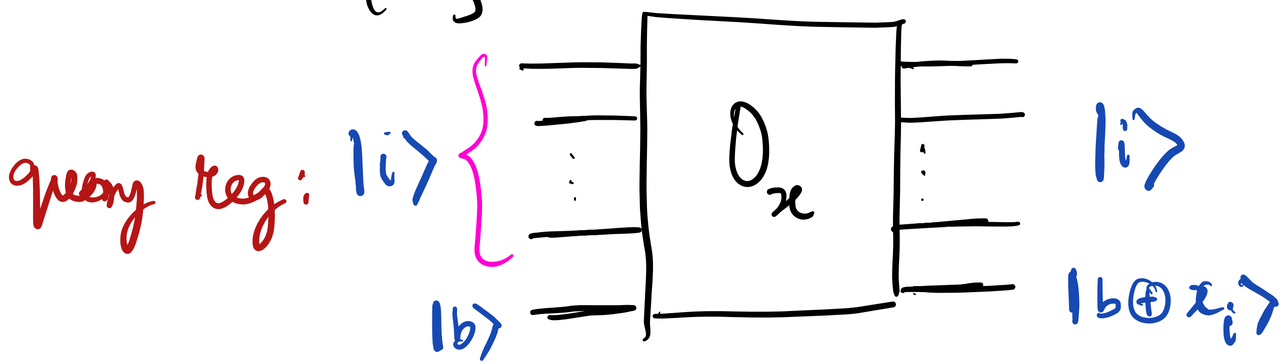

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$P(x)$ is a complex poly of deg $\leq t$

ϵ-Approximating Polynomials

$$\mathcal{M} \; U_t \, O_x \, U_{t-1} \cdots O_x \, U_2 \, O_x \, U_1 \, O_x \, U_0 \, |0\rangle$$

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$p(x)$ is a complex poly of $\deg \le t$

Corollary: $\Pr\left[\, |\psi_t\rangle \text{ on } \mathcal{M} \text{ outputs } 1 \,\right] = A(x)$

where $A \in \mathbb{R}[x_1, \ldots, x_n]$ of $\deg \le 2t$.

Optimality of Grover Search.

Theorem (Beals-Buhrman-Cleve-Mosca-de Wolf '98)

If f can be computed by a quantum query algorithm of cost t, then $\exists\, A \in \mathbb{R}[x_1 \cdots x_n]$ of degree $\leq 2t$, s.t. $\|f - A\|_\infty \leq \epsilon$, i.e. $\frac{\deg_\epsilon(f)}{2} \leq Q_\epsilon(f)$

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Notation

$$\widetilde{\deg}(f) := \deg_{1/3}(f).$$

Quantum-Classical Query Equivalence

Theorem:

For every total Boolean $f : \{0,1\}^n \to \{0,1\}$,

$$Q_\epsilon^{dt}(f) \leq R_\epsilon^{dt}(f) \leq D^{dt}(f) \underbrace{\leq}_{\text{BBCMdW}} bs^3(f) \underbrace{\leq}_{\text{NS}} \widetilde{deg}^6(f) \underbrace{\leq}_{\text{BBCMdW}} \left(Q_\epsilon^{dt}(f)\right)^6$$

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BBCMdW NS BBCMdW

$$\Downarrow$$

$\forall$ total Bool. f :

$$\deg(f) \le D^{dt}(f) \le \widetilde{\deg}^6(f)$$

(Purely Algebraic-Analytic Result!

Quantum-Classical Query Equivalence

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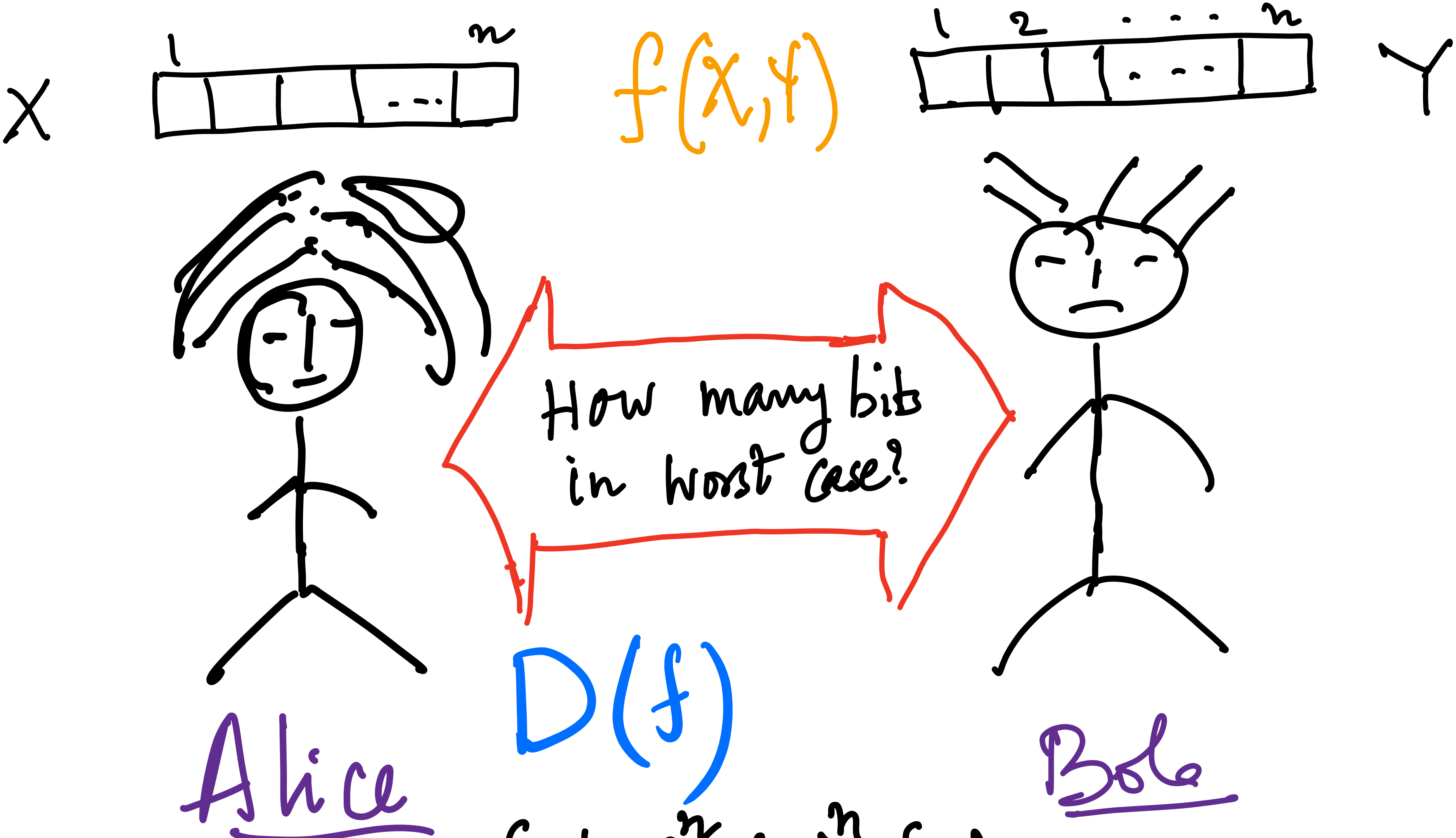
Theorem: (Aaronson-Ben-David-Kothari-Rao-Tal'21)

$$Q_\epsilon^{dt}(f) \le D^{dt}(f) \le (Q_\epsilon^{dt}(f))^4$$

$$\deg(f) \le (\widetilde{\deg}(f))^2.$$

Builds on Huang's BREAKTHROUGH!

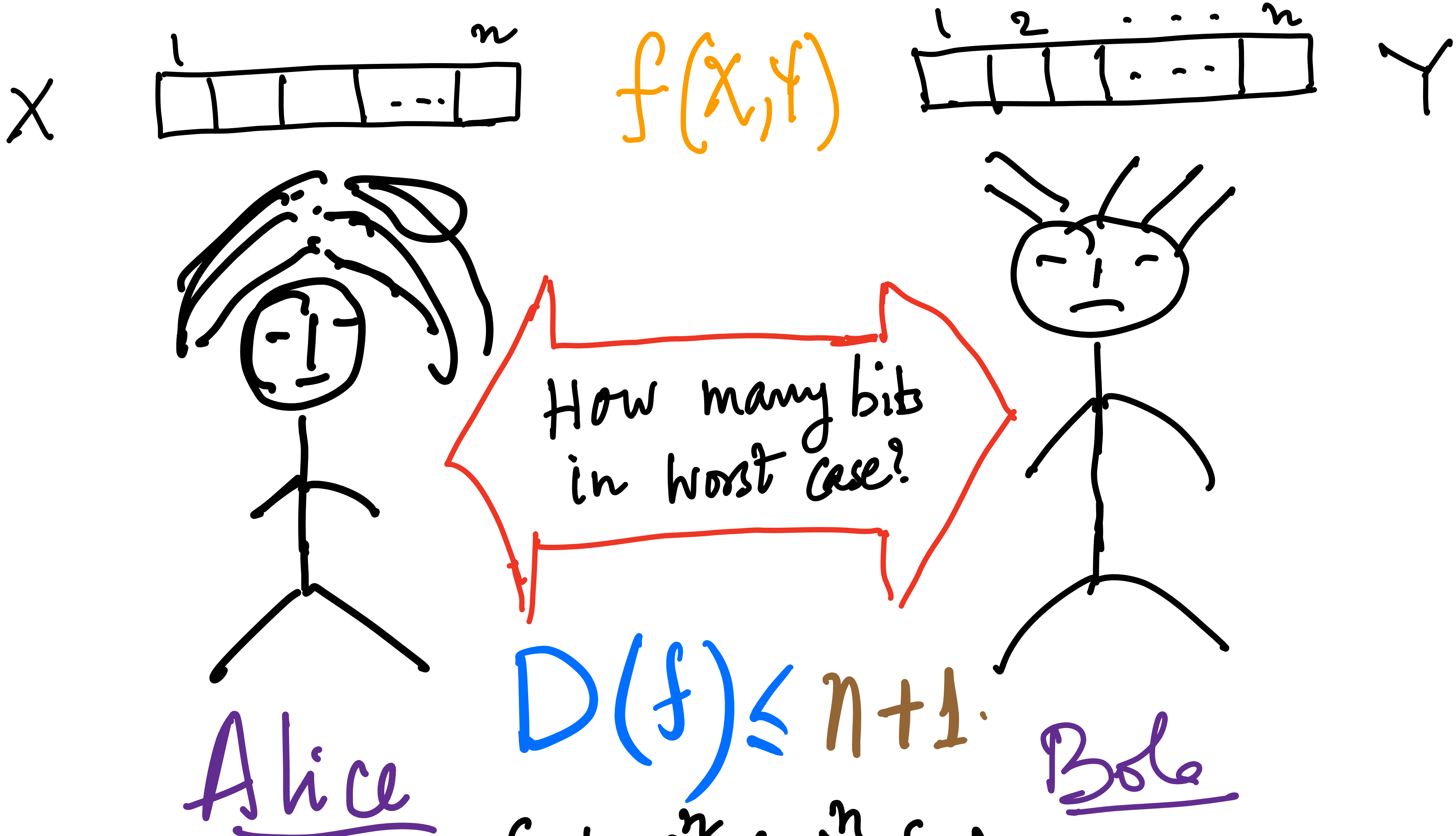
Simple Model - II



$D(f)$

$f: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$

Simple Model - II



$$D(f) \leq n+1$$

$$f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$$

Easy Functions.

Examples:

PARITY, MAJORITY, SYMM....

Communication Complexity

Unexpected Connections.

(i) VLSI
(ii) Distributed Computing
(iii) Data Structures
(iv) Streaming Algorithms
(v) Game Theory
(vi) Extension Complexity
⋮

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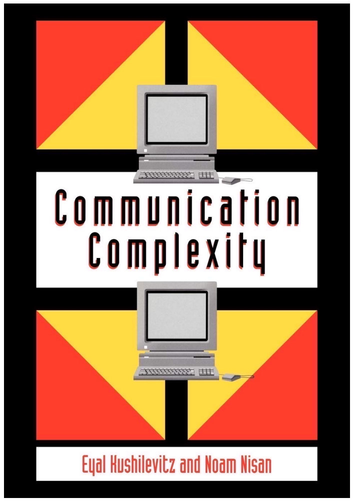

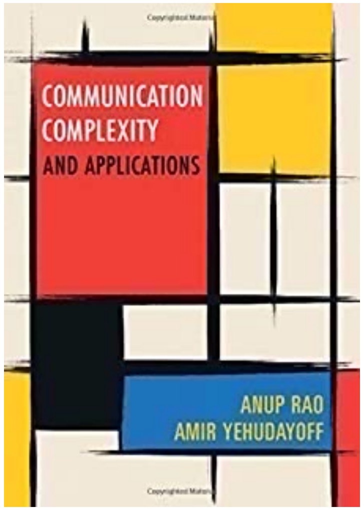

Rich mathematical theory

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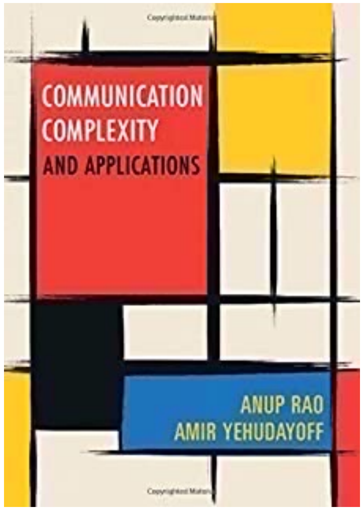
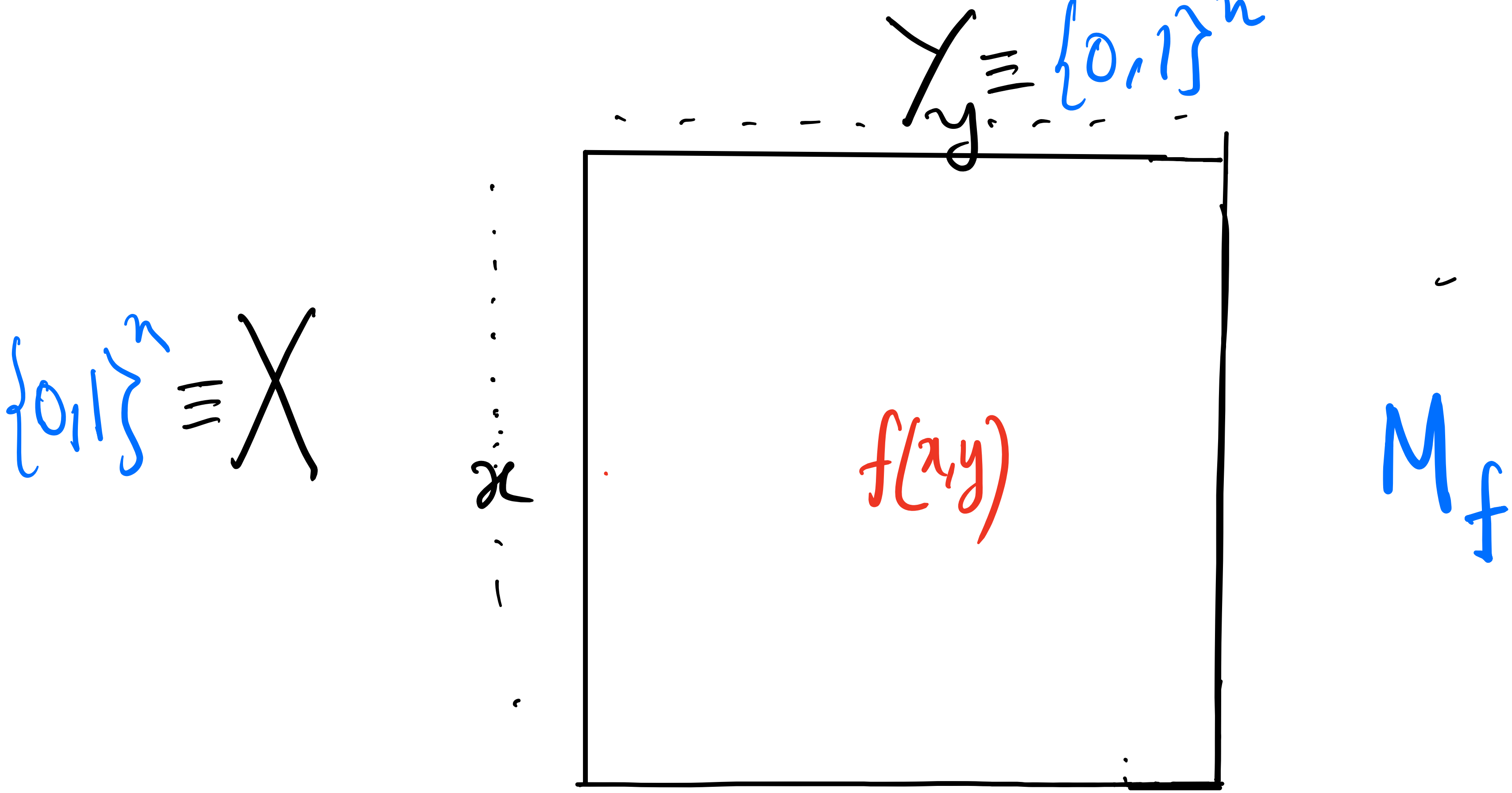


Rich mathematical theory

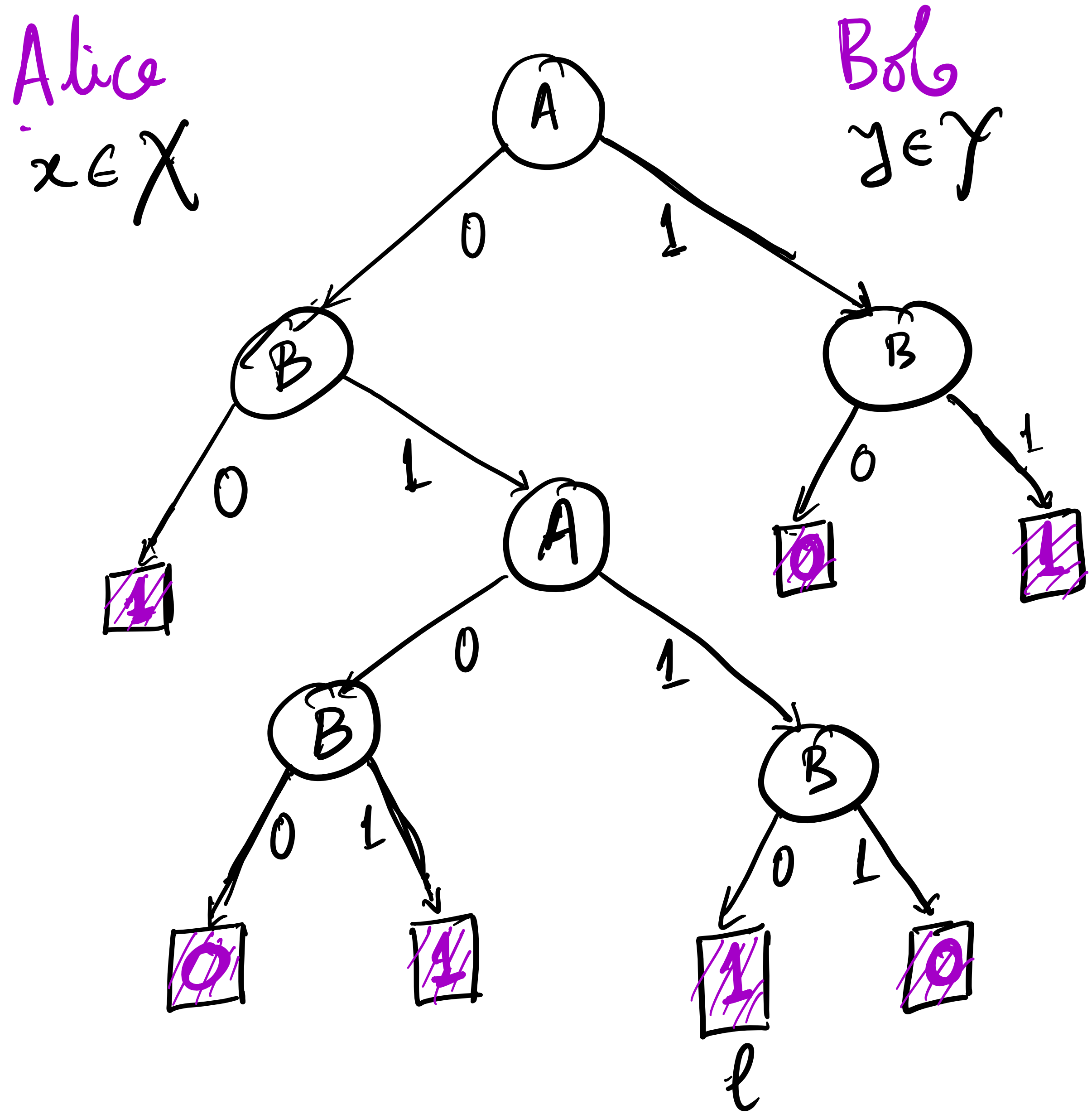
Many Open Problems!

The Communication Matrix



Fact: $\mathrm{Rank}(M_f) \leq 2^{D^{cc}(f)}$

A Communication Protocol



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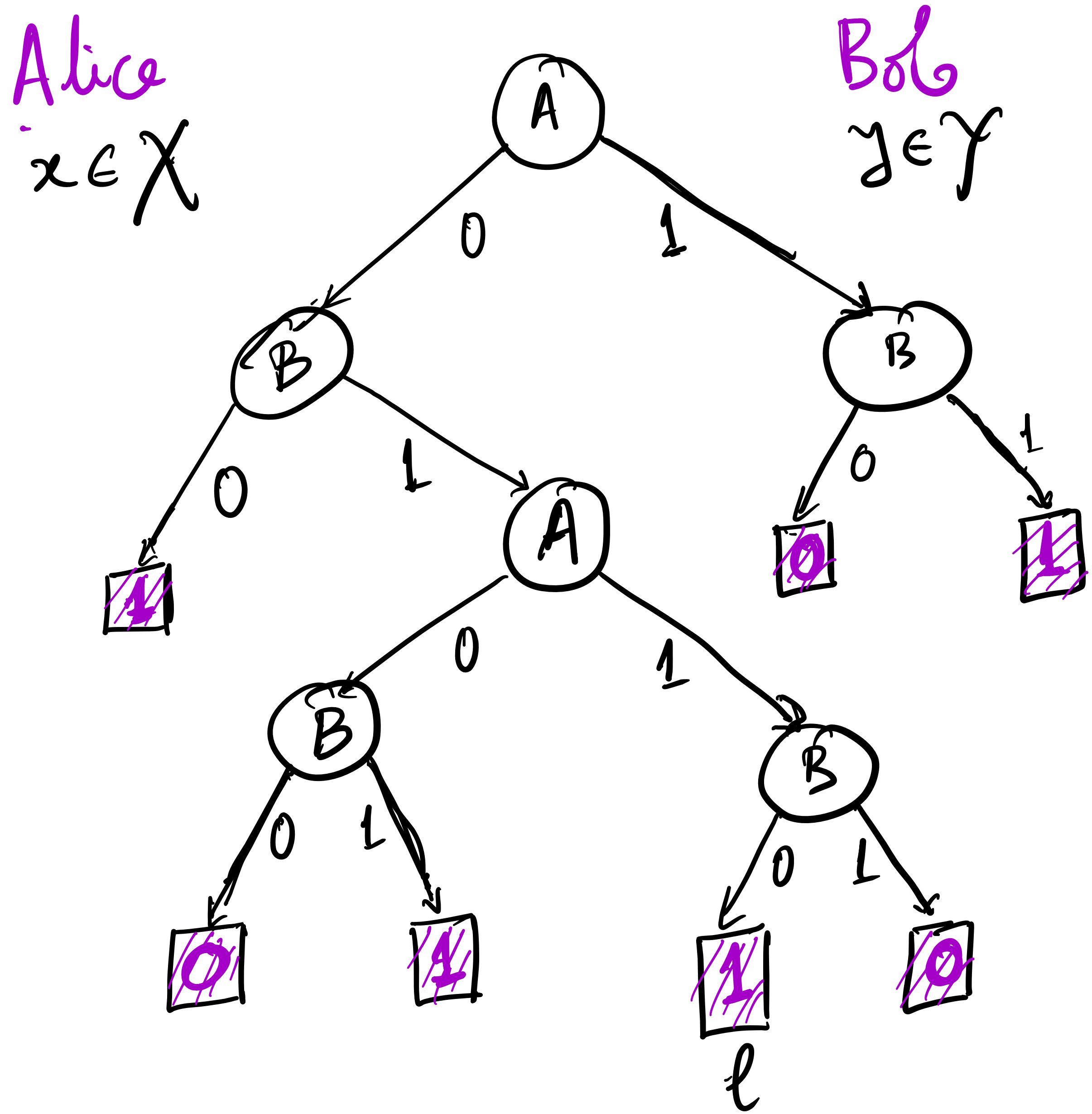


(x,y) accepted

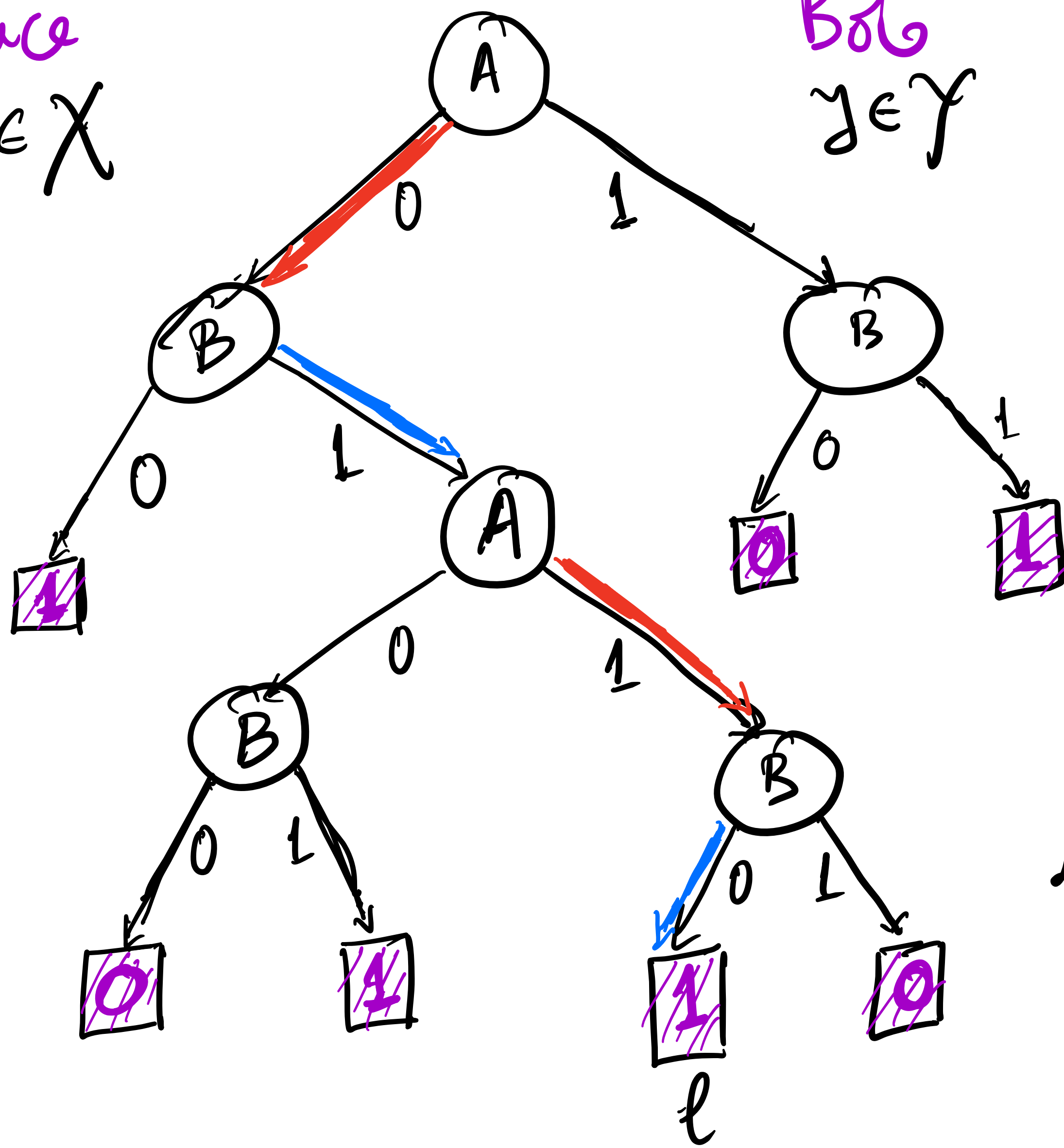
$\Longleftrightarrow$

(x,y) reaches 1-leaf

A Communication Protocol

Alice $x \in X$

Bob $y \in Y$



(x,y) accepted

$\Longleftrightarrow$

(x,y) reaches 1-leaf

Inputs reach ℓ

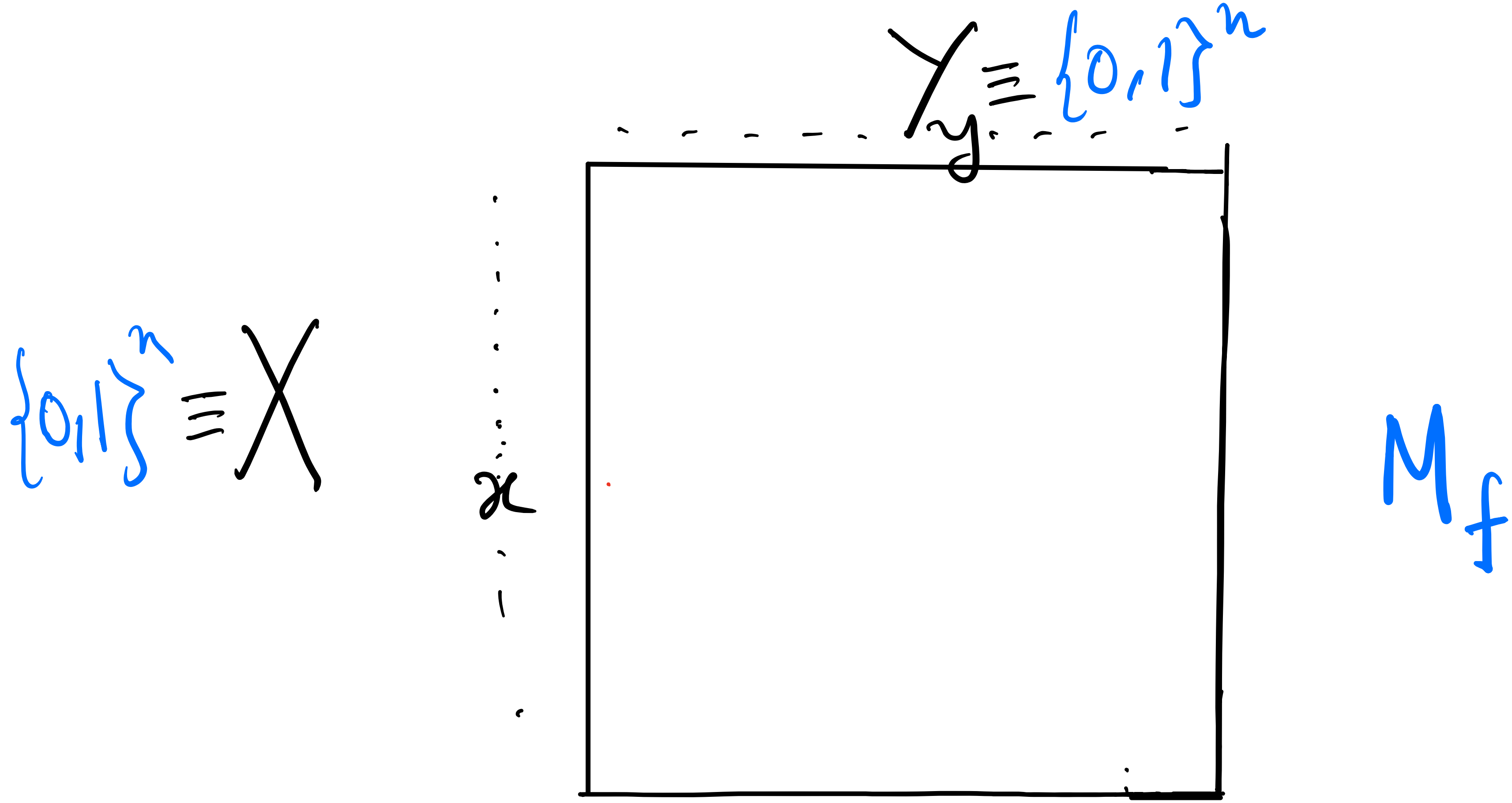
$=$

$\{x : \text{Alice answers red}\}$

$\times$

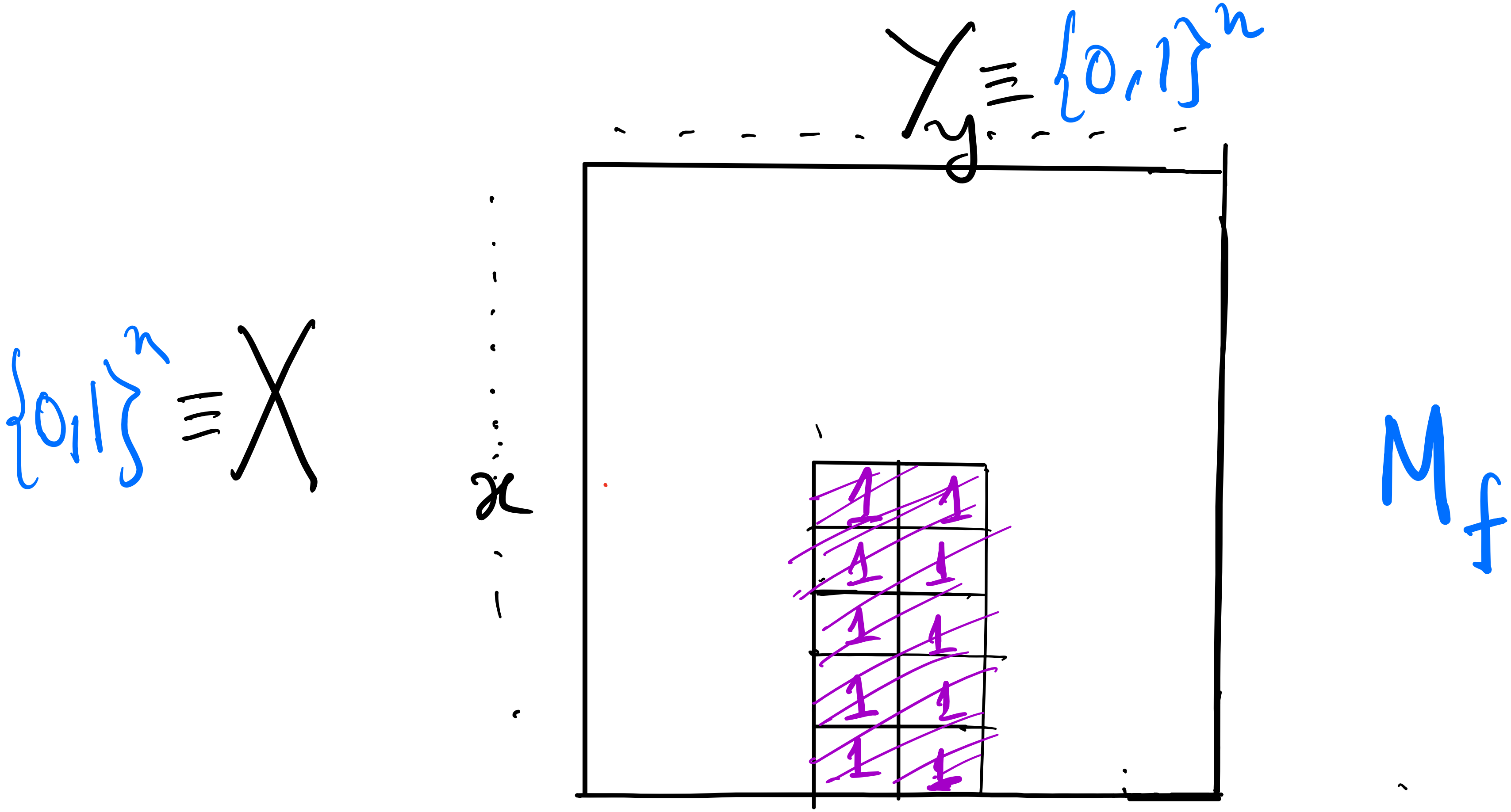
$\{y : \text{Bob answers blue}\}$

Rank Implications



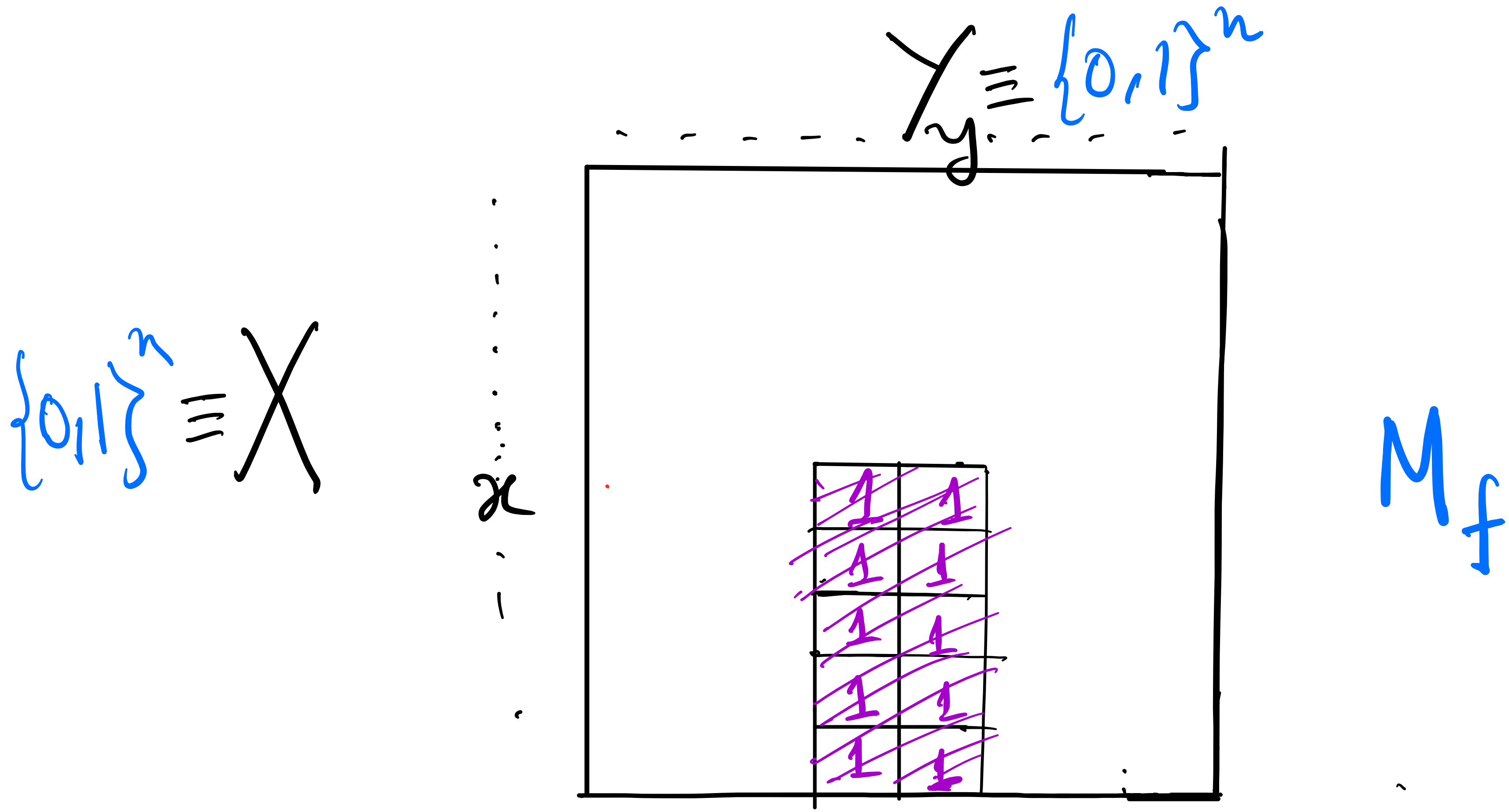
<u>Truth table of function</u>

Rank Implications



Leaf ℓ contributes rank 1

Rank Implications



$$\mathrm{Rank}(M_f) \leq \sum_{\ell \in \mathrm{Leaf}_1(\pi)} \mathrm{Rank}(M_f[X_\ell \times Y_\ell]) \leq 2^{\mathrm{cost}(\pi)}$$

Mehlhorn & Schmidt '82

$$\log(\mathrm{rank}(M_f)) \leq D^{cc}(f)$$

DT - analogy : $\deg(f) \leq D^{dt}(f)$

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Krause:
$$\log(\widetilde{\text{rank}}(M_f)) \leq R^{cc}(f)$$

DT - analogy: $\widetilde{\deg}(f) \leq R^{dt}(f)$

Kremer:
$$\log(\widetilde{\text{rank}}(M_f)) \leq Q^{cc}(f)$$

DT - analogy: $\widetilde{\deg}(f) \leq Q^{dt}(f)$

Upper Bounds: Analogy...

Log-Rank $D^{dt}(f) \leq \deg^{b}(f)$

Conjecture: (Lovasz - Saks '88)

$$\forall \text{ total } f: \; D^{cc}(f) \leq \left(\log(\operatorname{rank}(M_f))\right)^{O(1)}$$

Log-Approx-Rank $R^{dt}(f) \leq \widetilde{\deg}^{b}(f)$

Conjecture: (Lee-Shraibman '07)

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$$\Uparrow$$

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Conjectured Equivalence.

Log-Approx-Rank Conjecture: (Lee-Shraibman '07)

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$$\Downarrow$$

$$\boxed{\begin{array}{l}\textbf{Log-Equivalence Conjecture: } \text{(Shi-Zhu '07).} \\ \forall \text{ total } f: \quad Q^{cc}(f) = \left(R^{cc}(f)\right)^{O(1)}\end{array}}$$

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Major Open Problem!

Lowering Our Aim

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Major Open Problem

Question: Can we prove it for a broad and natural class of functions?

Composed Functions

Commonly used functions

(a) Set-Disjointness

(b) Inner-Product

(c) Equality

Composed Functions

Commonly used functions

(a) Set-Disjointness $\quad NOR_n \circ AND_2$

(b) Inner-Product $\quad \oplus_n \circ AND_2$

(c) Equality $\quad AND_n \circ \oplus_2$

Composed Functions

Commonly used functions

(a) Set-Disjointness $\quad NOR_n \circ AND_2$

(b) Inner-Product $\quad \oplus_n \circ AND_2$

$\Big\}$ AND functions

(c) Equality $\quad AND_n \circ \oplus_2$

$\Big\}$ XOR functions

Composed Functions

Commonly used functions

(a) Set-Disjointness $\quad \mathrm{NOR}_n \circ \mathrm{AND}_2$

(b) Inner-Product $\quad \oplus_n \circ \mathrm{AND}_2$

AND functions

(c) Equality $\quad \mathrm{AND}_n \circ \oplus_2$

XOR functions

Question: Can we prove equivalence for the class of AND and XOR functions?

Results: AND and XOR Functions

Log-Approx-Rank Conjecture: (Lee-Shraibman '07)

$\forall$ total f: $R^{cc}(f) \leq \left(\log\left(\widetilde{\mathrm{rank}}(M_f)\right)\right)^{O(1)}$

$\Downarrow$

Log-Equivalence Conjecture: (Shi-Zhu '07).

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Results: AND and XOR Functions

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FALSE for XOR fns.

C-Mande-Sherif '2019

$\Downarrow$

Log-Equivalence

Conjecture: (Shi-Zhu '07).

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Results: AND and XOR Functions

True for AND functions!

Bhattacharya - Byramji - C - Dahiya - Lovett '26

Log-Approx-Rank

Conjecture: (Lee-Shraibman '07)

$\forall$ total f: $R^{cc}(f) \le \left(\log\left(\widetilde{\text{rank}}(M_f)\right)\right)^{O(1)}$

FALSE for XOR fns.

C - Mande - Sherif '2019

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Open:

Is LEC true for XOR fns?

Quantum's Advantage Over Determinism

Question: Quantum vs. Determinism for AND-fns?

No super-poly advantage was known!

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No super-poly advantage was known!

Recall:

$$Q^{dt}(f) \le D^{dt}(f) \le \deg^3(f) \le \widetilde{\deg}^6(f) \le \left(Q^{dt}(f)\right)^6$$

Midrijanis

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Midrijanis

$$\text{LRC} \iff D^{cc}(f \circ \wedge_2) = \left(\log \operatorname{rank}(f \circ \wedge_2)\right)^{O(1)}$$

Rank of AND-functions

Let, $f = \sum_{S \subseteq [n]} c_S^f \prod_{i \in S} x_i$; $x_i \in \{0,1\}$

De Morgan basis

unique representation

$$\text{Spar}(f) = \left| \{ S \subseteq [n] \mid c_S^f \neq 0 \} \right|$$

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Fact: $\text{Rank}(M_{f \circ \wedge_2}) = \text{Spar}(f)$.

Log-Rank Holds for AND-fns

Theorem: (Knop-Lovett-McGuire-Yuan'2021)

$$D^{cc}(f \circ \wedge_2) = O\left(\log^5(\text{sparsity}(f)) \cdot \log n\right)$$

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KLMY'21 BBCDL'26

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KLMY'21

BBCDL'26

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Basic Result on Polynomial

Corollary: (Conjectured by KMLZ'21)

$\forall$ total f,

$$\log(\mathrm{spar}(f))^{\Omega(1)} \underset{\text{BBCDL-26}}{=} \log\left(\widetilde{\mathrm{rank}}\left(M_{f\circ\wedge_2}\right)\right) \underset{\text{Folklore}}{=} O\left(\log\left(\widetilde{\mathrm{spar}}(f)\right)\right)$$

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Conclusion: Approx doesn't help reduce Sparsity much!

Sparsity analogue of $\deg(f) = O(\widetilde{\deg}(f))^2$

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First proved in C-Dahiya-Lovett'25.

Why

Approximation Doesn't Reduce Sparsity Much?

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Take Simple Example of OR_n

Approximate Sparsity of OR

$$\mathrm{OR}(x_1 \cdots x_n) := 1 - \prod_{i=1}^{n} (1 - x_i)$$

$\mathrm{spar}(\mathrm{OR}_n) = 2^n - 1$

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Question: How do we prove nothing better is possible?

Random Restriction Argument

$OR(x) \approx_\epsilon P(x)$, $\mathrm{spar}(P) = s$.

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For each $i \in [n]$, $\begin{cases} \text{w.p. } \frac{1}{2} \text{ set } x_i = 0, \\ \frac{1}{2} \text{ set } x_i = * \text{ (free)}. \end{cases}$

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$M := x_{i_1} x_{i_2} \cdots x_{i_t}$ be a monomial of degree t.

$\Pr[M_\rho \neq 0] \leq \frac{1}{2^t}$, $\forall \rho: OR_\rho(x) \approx_\epsilon P_\rho(x)$

$\Pr[\deg(P_\rho) > t] \leq \frac{s}{2^t}$

If $s < \frac{1}{10} \cdot 2^{c\sqrt{n}}$,

$$\Pr\left[\deg(P_\rho) \geqslant c\sqrt{n}\right] < \frac{1}{10}$$

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Nisan-Szegedy

$\widetilde{\deg}(OR_{\frac{n}{2}}) \geq c\sqrt{n}$.

Fact: $\log \widetilde{\text{spar}}(OR_n) = \Theta(\sqrt{n})$.

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Contradiction!

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Semi-adaptive Max-degree restrictions for $f:\{0,1\}^n \to \mathbb{R}$.

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$\exists\, V \subseteq [n]$, s.t.

$\forall \rho \in \{0,*\}^V$, $\exists$ a fixing σ_ρ of vars in $[n]\setminus V$

s.t. $f_{\rho\circ\sigma_\rho}$ has full degree, i.e. $\deg(f_{\rho\circ\sigma_\rho}) = |\mathrm{free}(\rho)|$

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OR$_n$: $V = [n]$

max-degree restriction of depth n

Utility of Such Restrictions.

Observation:

If f is a total Boolean function that admits a semi-adaptive restriction tree of depth d, then

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Same argument as for OR_n, Nisan-Szegedy replaced by A-BDK-R-T $\left(\deg(h) = O(\widetilde{\deg}^2(h))\right)$

Structure of Non-Sparse Polynomials

Main Lemma: (Bhattacharya-Byramji-C-Dahiya-Lovett '26).

Let $Q : \{0,1\}^n \to \mathbb{R}$ have $\mathrm{Spar}(Q) = s$. Then, $\exists$ a semi-adaptive max restriction tree of depth $\Omega\left(\frac{\log s}{\log n}\right)$ for Q.

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$$\Downarrow$$

$\forall$ total Boolean f:

$$\log \widetilde{\mathrm{spar}}(f) = \tilde{\Omega}\left(\sqrt{\log \mathrm{spar}(f)}\right)$$

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How Non-Sparsity Yields Semi-Adaptive Restrictions!

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Sauer-Shelah Lemma $\Rightarrow$ $|\mathcal{M}| = O(n^{|V|})$

$$|V| = \Omega\left(\frac{\log s}{\log n}\right).$$

Consider any restriction $\rho \in \{0, *\}^V$

V $\quad\quad\quad\quad\quad$ $[n] - V$

$\boxed{0 * 0 0 * 0 0 * 0 0}\ \boxed{}$

$T = \{ * \quad * \quad * \}$

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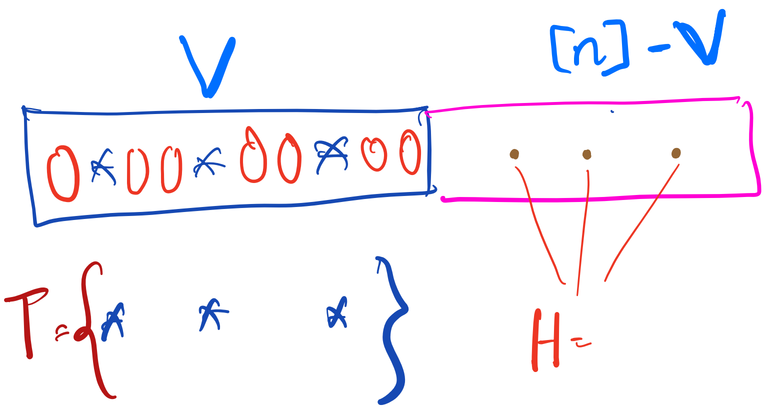

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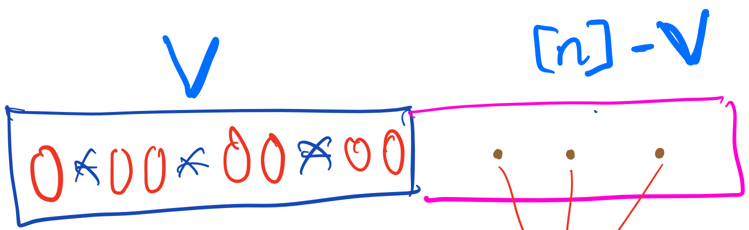

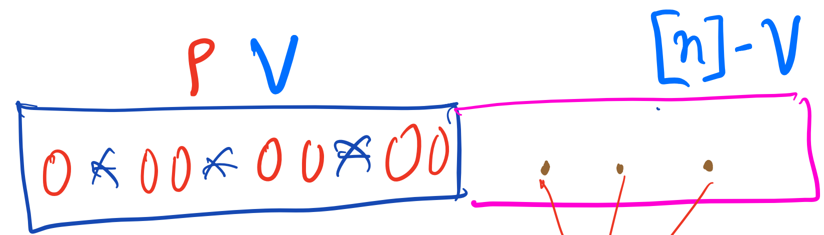

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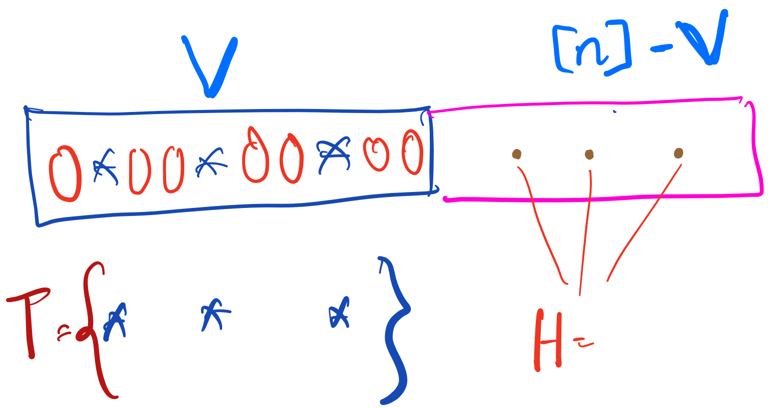

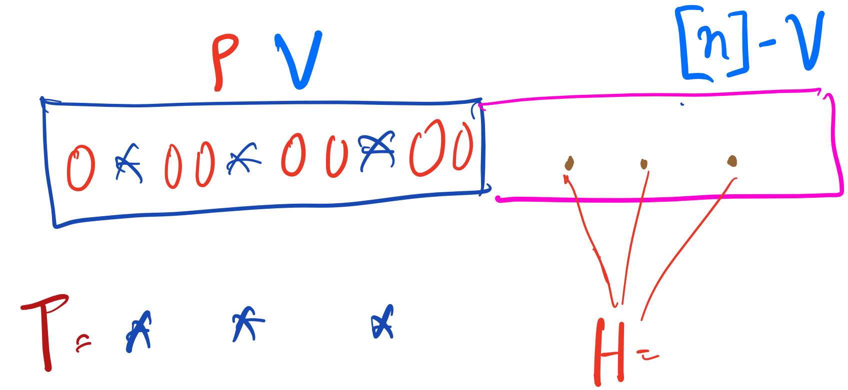

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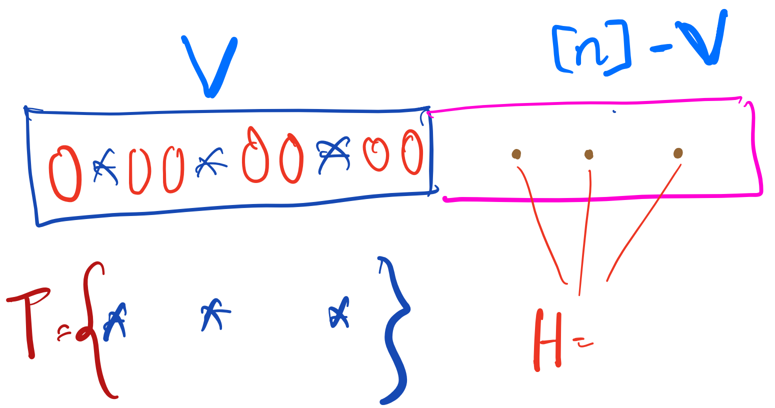

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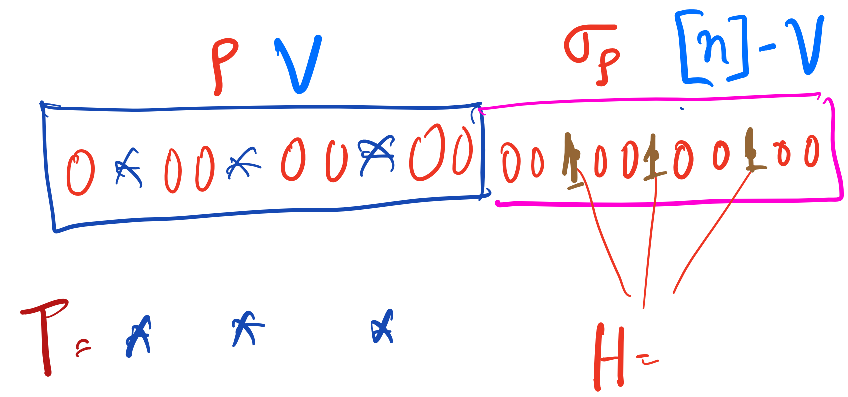

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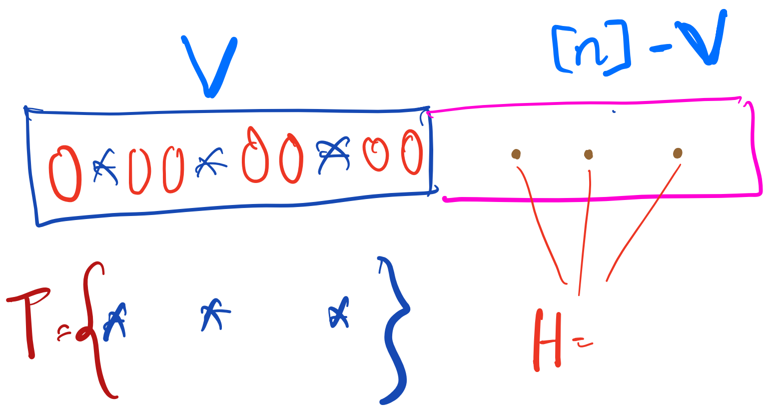

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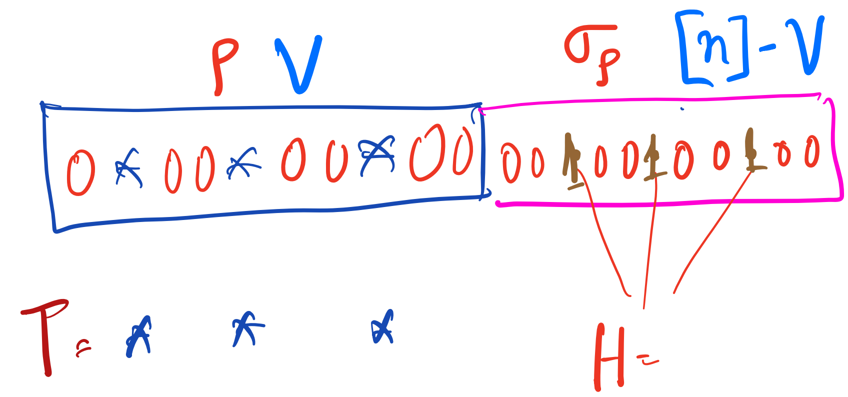

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$\Rightarrow$ monomial $\prod_{i \in T} x_i$ appears in $Q_{\rho\circ\sigma_\rho}$

Sparsity to Rank

$\forall$ total Boolean f:

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$$\Big\Downarrow \text{Lifts}$$

$$\|$$

$$\log\left(\mathrm{rank}(f \circ \wedge_2)\right) = \tilde{O}\left(\log^4 \widetilde{\mathrm{rank}}(f \circ \wedge_2)\right)$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\Theta\left(D^{cc}(f \circ \wedge_2)\right) \qquad \frac{\text{Quantum-}}{\text{classical}} \qquad O\left(Q^{cc}(f \circ \wedge_2)\right)^4$$

Equivalence

Sparsity to Rank

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$\|$ $\quad\downarrow$ Lifts $\quad$ Requires more Fourier-theoretic ideas

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Next!

Protocol Rank Non-Equivalence

Theorem (C-Mande-Sherif '2019)

There is a function $f : \{0,1\}^n \to \{0,1\}$, s.t.,

$$\boxed{\begin{aligned} \widetilde{\mathrm{rank}}(f \circ \oplus_2) &= O(n^2) \\ R_{1/3}(f \circ \oplus_2) &= \Theta(\sqrt{n}) \end{aligned}}$$

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Exponential Separation of $\log \widetilde{\text{rank}}(F)$ vs. $R_{1/3}(F)$.

Protocol Rank Non-Equivalence

Theorem: (Anshu - Goud Boddu - Touchette 2019
Sinha - de Wolf 2019)

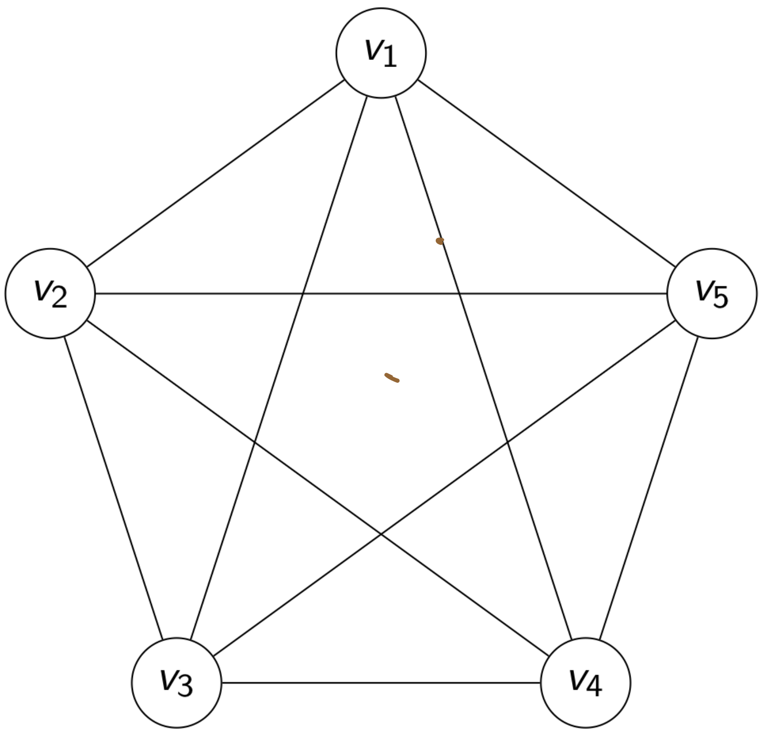
For the same function $f : \{0,1\}^n \to \{0,1\}$

$$\boxed{\begin{aligned} \widetilde{\text{rank}}(f \circ \oplus_2) &= O(n^2) \\ Q_{1/3}(f \circ \oplus_2) &= \Omega(n^{1/6}) \end{aligned}}$$

Exponential Separation of $\log \widetilde{\text{rank}}(F)$ vs. $Q_{1/3}(F)$.

The Function - Query

$$\text{SINK}: \{0,1\}^{\binom{n}{2}} \to \{0,1\}$$



The Function - Query

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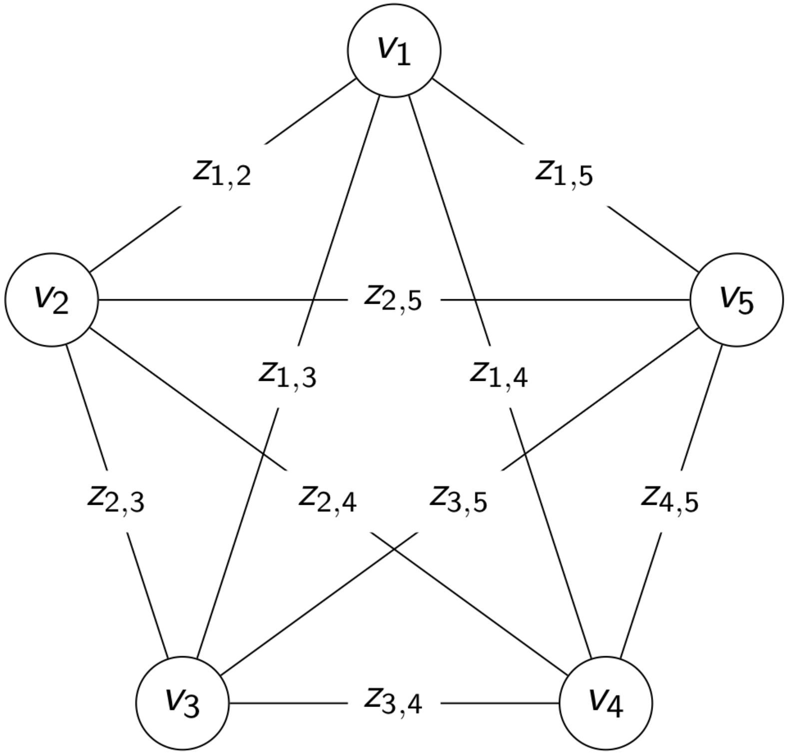

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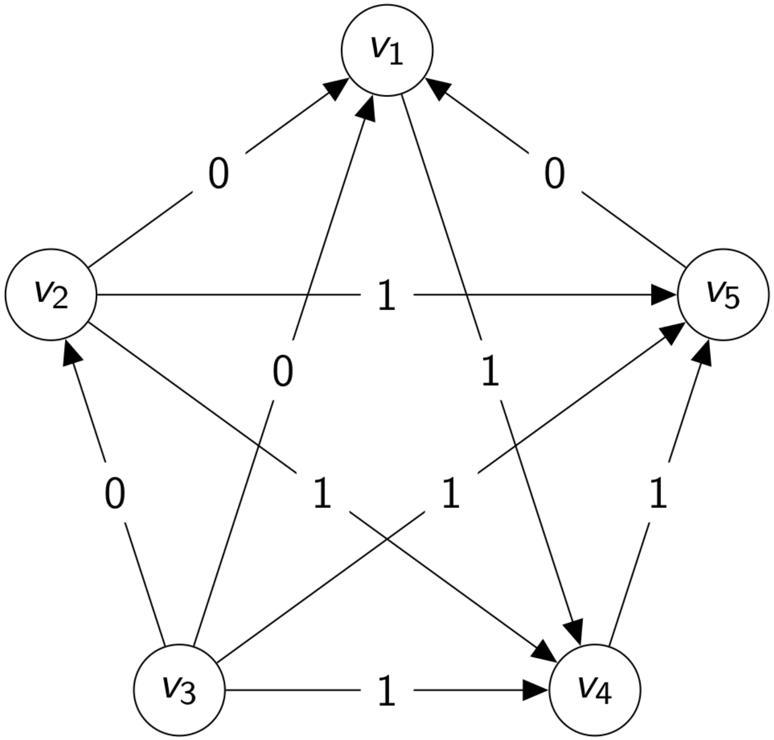
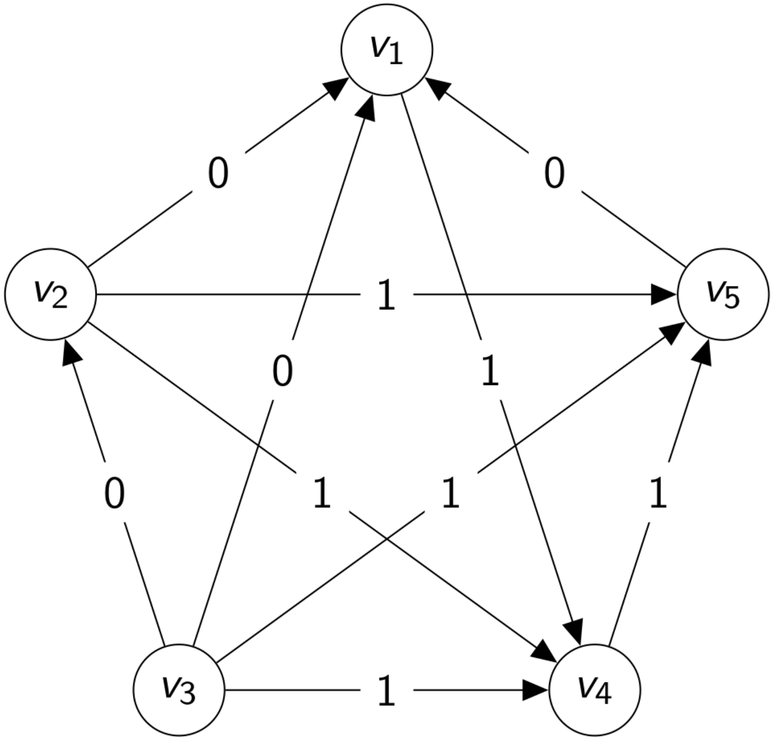


The Function - Query

$$\text{SINK}: \{0,1\}^{\binom{n}{2}} \to \{0,1\}$$



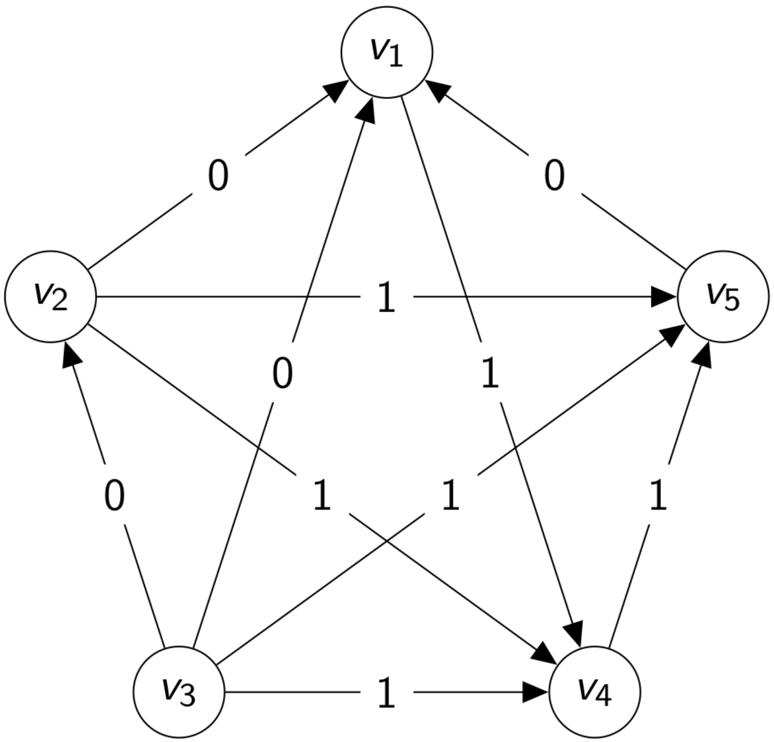
$$\text{SINK}(z) = 1$$

$$\Updownarrow$$

$\exists v_i : v_i$ is a sink

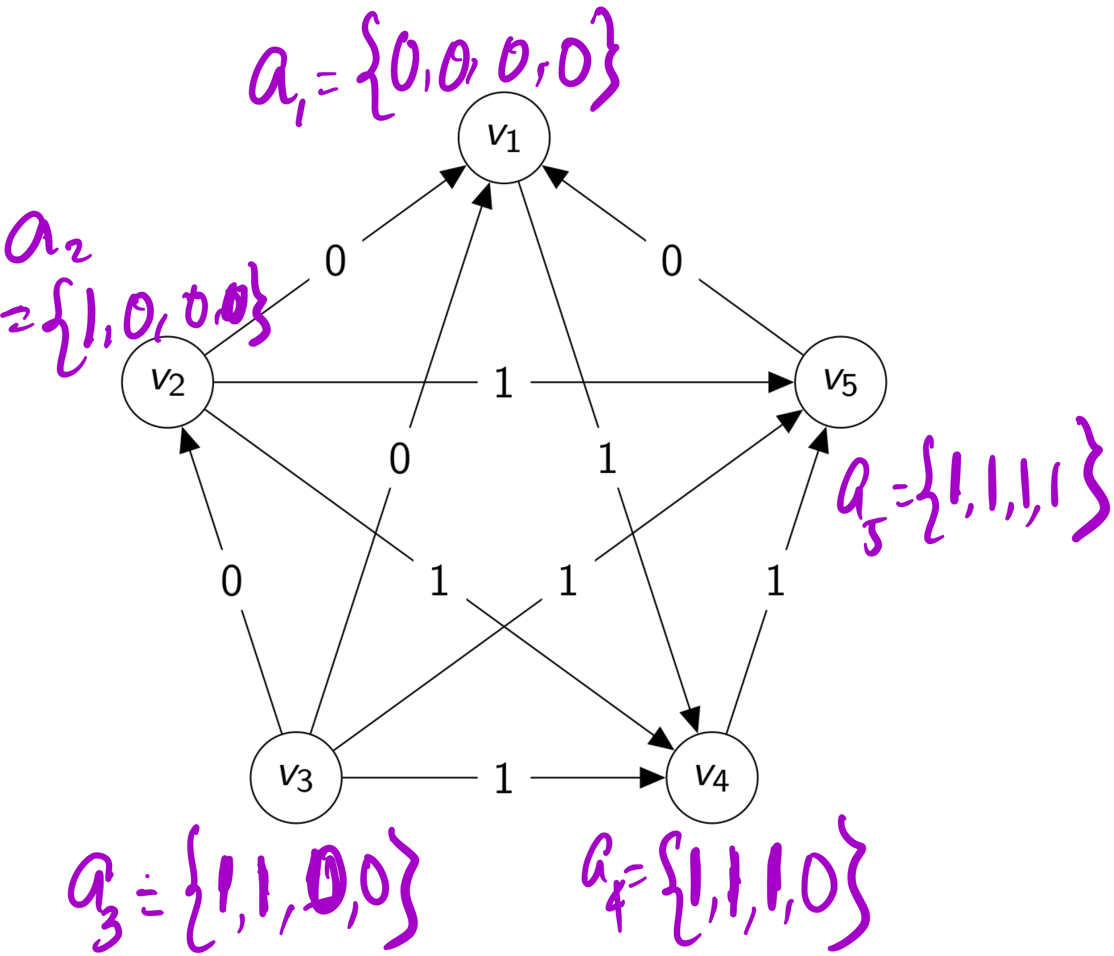
Low Approximate-Rank

Theorem: $\mathrm{rank}_{\frac{1}{3}}(\mathrm{SINK} \circ \mathrm{XOR}) = O(n^4) = O(n^2)$



Low Approximate-Rank

Theorem: $\mathrm{rank}_{\frac{1}{3}}(\mathrm{SINK} \circ \mathrm{XOR}) = O(n^4) = O(n^2)$



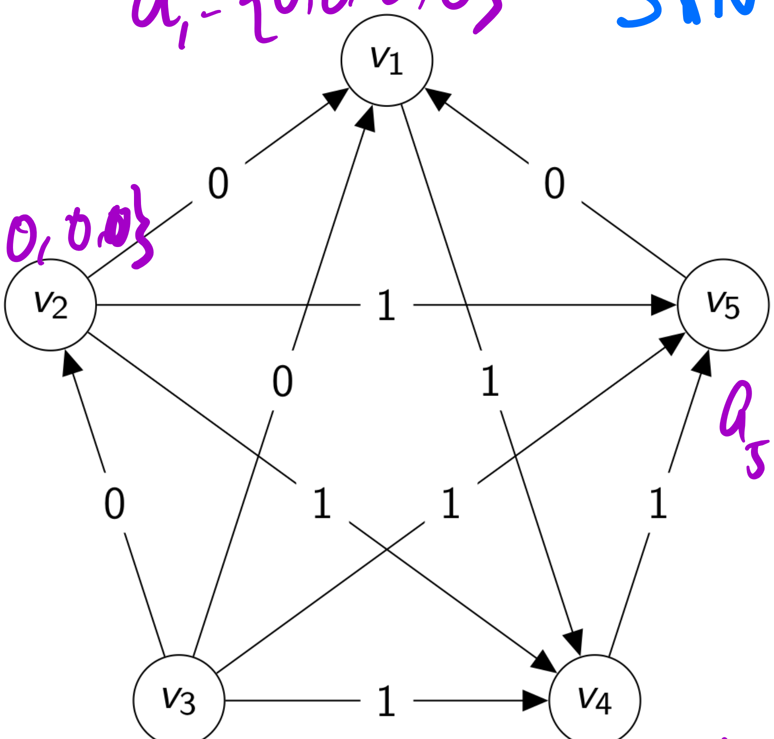
Low Approximate-Rank

Theorem: $\mathrm{rank}_{\frac{1}{3}}(\mathrm{SINK} \circ \mathrm{XOR}) = O(n^4) = O(n^2)$

$$\mathrm{SINK} \circ \mathrm{XOR} = \sum_i \mathrm{EQ}\left((x_{v_i} \oplus y_{v_i}), a_i \right)$$

$a_1 = \{0, 0, 0, 0\}$

$a_2 = \{1, 0, 0, 0\}$

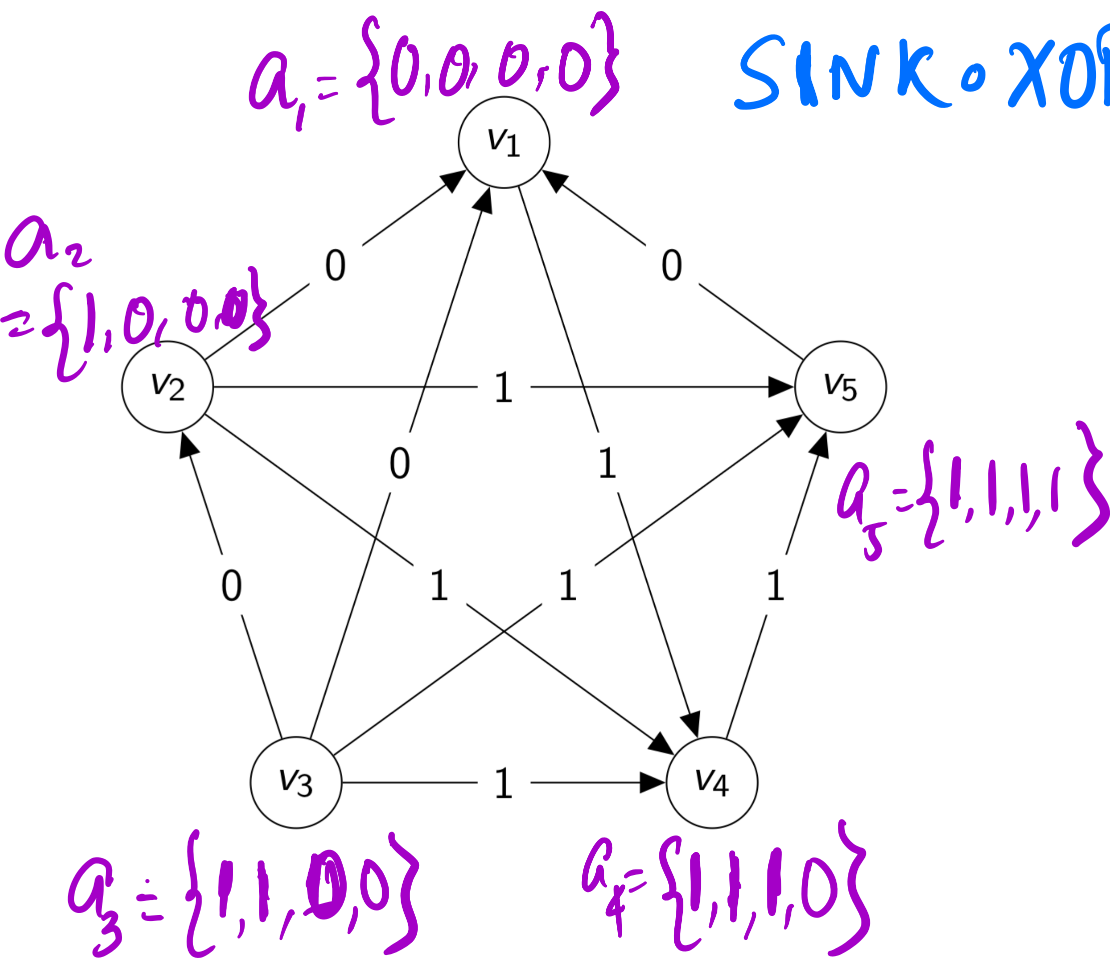


$a_5 = \{1, 1, 1, 1\}$

$a_3 = \{1, 1, 0, 0\}$ $\quad$ $a_4 = \{1, 1, 1, 0\}$

Low Approximate-Rank

Theorem: $\mathrm{rank}_{\frac{1}{3}}(\mathrm{SINK}\circ\mathrm{XOR}) = O(n^4) = O(n^2)$



$$\mathrm{SINK}\circ\mathrm{XOR} = \sum_i EQ\left((x_{v_i} \oplus y_{v_i}), a_i\right)$$

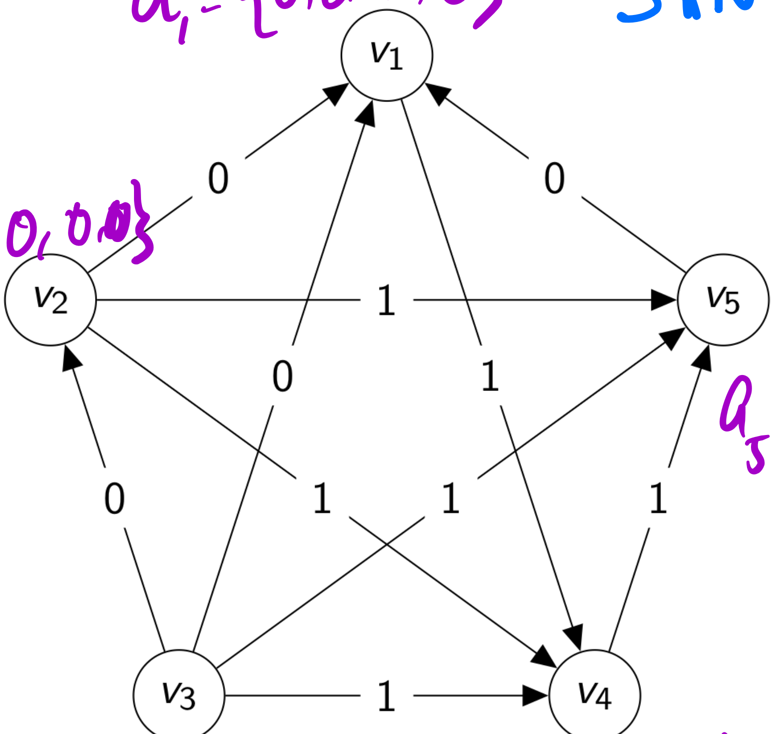
$$= \sum_i EQ\left(x_{v_i}, (y_{v_i} \oplus a_i)\right)$$

Low Approximate-Rank

Theorem: $\mathrm{rank}_{\frac{1}{3}}(\mathrm{SINK} \circ \mathrm{XOR}) = O(n^4) = O(n^2)$

$a_1 = \{0, 0, 0, 0\}$

$a_2 = \{1, 0, 0, 0\}$



$a_5 = \{1, 1, 1, 1\}$

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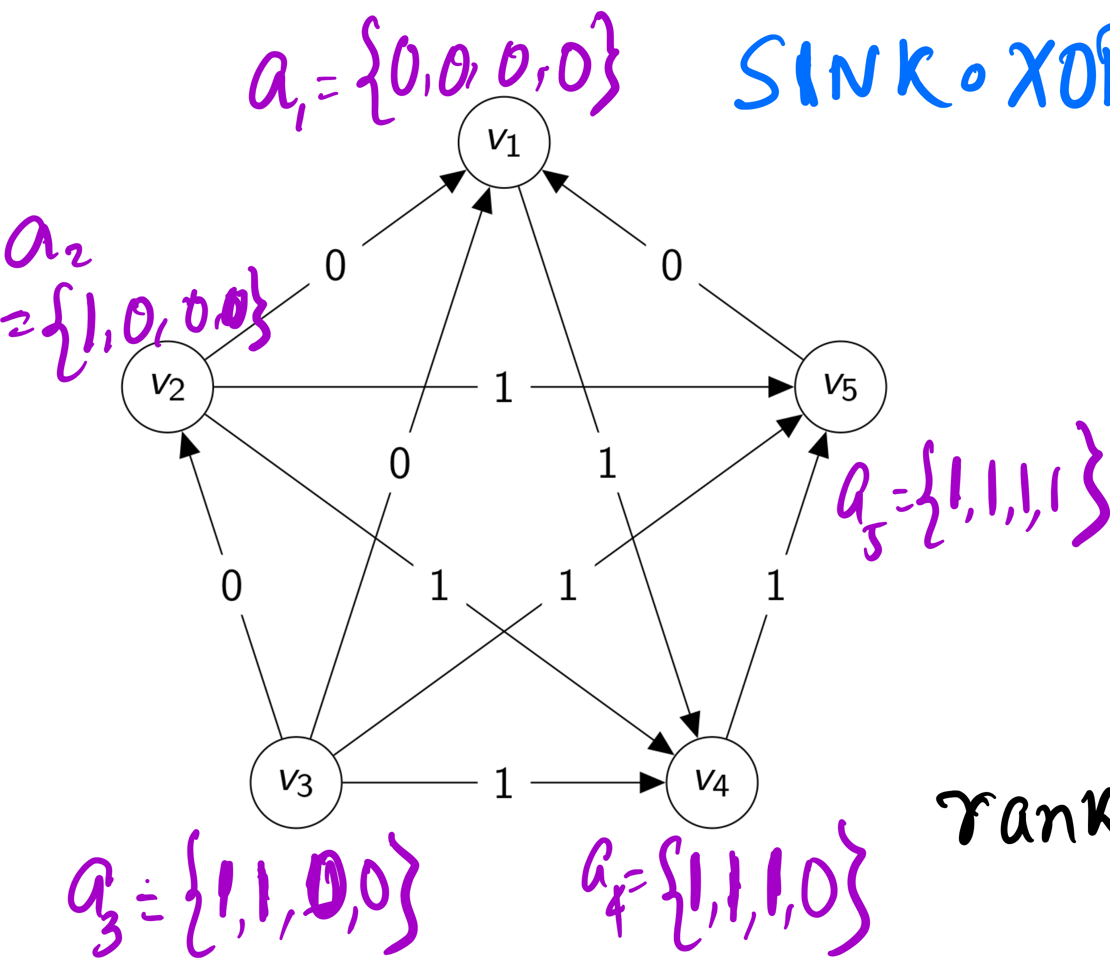
$$\mathrm{SINK} \circ \mathrm{XOR} = \sum_i EQ\left(\left(x_{v_i} \oplus y_{v_i}\right), a_i\right)$$

$$= \sum_i EQ\left(x_{v_i}, \left(y_{v_i} \oplus a_i\right)\right)$$

$$= \sum_i EQ\left(x_{v_i}, y'_{v_i}\right)$$

Low Approximate-Rank

Theorem: $\mathrm{rank}_{\frac{1}{3}}(\mathrm{SINK}\circ\mathrm{XOR}) = O(m^4) = O(n^2)$



$$\mathrm{SINK}\circ\mathrm{XOR} = \sum_i EQ\left(\left(x_{v_i}\oplus y_{v_i}\right), a_i\right)$$

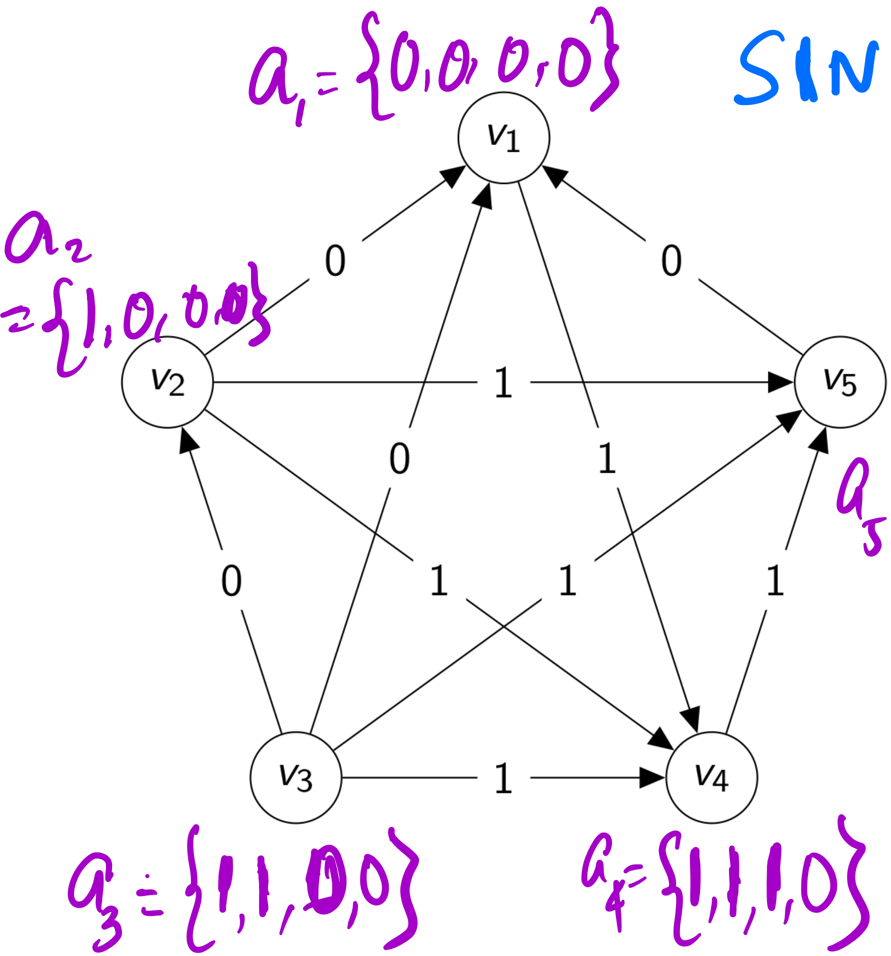
$$= \sum_i EQ\left(x_{v_i}, \left(y_{v_i}\oplus a_i\right)\right)$$

$$= \sum_i EQ\left(x_{v_i}, y'_{v_i}\right)$$

$$\mathrm{rank}_{\frac{1}{3}}(\mathrm{SINK}\circ\mathrm{XOR}) \leq m\cdot O(m^4)$$

Low Approximate-Rank

Theorem: $\text{rank}_{\frac{1}{3}}(\text{SINK} \circ \text{XOR}) = O(m^4) = O(n^2)$



$$\text{SINK} \circ \text{XOR} = \sum_i EQ\left(\left(x_{v_i} \oplus y_{v_i}\right), a_i\right)$$

$$= \sum_i EQ\left(x_{v_i}, \left(y_{v_i} \oplus a_i\right)\right)$$

$$= \sum_i EQ\left(x_{v_i}, y'_{v_i}\right)$$

$$\text{rank}^{+}_{\frac{1}{3}}(\text{SINK} \circ \text{XOR}) \leq m \cdot O(m^4) = O(n^{2.5})$$

non-negative-rank.

Our Results.

Log-Approx-Rank Conjecture: (Lee-Shraibman '07)

$$\forall \text{ total } f: \quad R^{cc}(f) \leq \left(\log\left(\widetilde{\mathrm{rank}}(M_f)\right)\right)^{O(1)}$$

FALSE for XOR fns.
C-Mande-Sherif '2019

$\Downarrow$

$$\boxed{\begin{array}{l}\text{Log-Equivalence Conjecture: (Shi-Zhu '07).}\\ \forall \text{ total } f: \quad Q^{cc}(f) = \left(R^{cc}(f)\right)^{O(1)}\end{array}}$$

True for AND functions!

Bhattacharya-Byramji-C-Dahiya-Lovett '26

Tantalizgly Open

a) QA for XOR functions?

Tantalizgly Open

a) QA for XOR functions?

b). Log-Rank-Conjecture for XOR fns?

Thank You!