Practice Problems: Total Dual Integrality and Total Unimodularity

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Submit the solutions of the questions marked $(\star)$ in PDF format generated using Latex by **March 20, 2026**.

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1. **$(\star)$** Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two matroids on the same ground set E, with rank functions r_1 and r_2 respectively.

   The matroid intersection polytope is defined as the convex hull of the incidence vectors of common independent sets:

   $$P = \text{conv}\{\mathbf{1}_I : I \in \mathcal{I}_1 \cap \mathcal{I}_2\}.$$

   Shaw that

   $$P = \left\{ x \in \mathbb{R}^E_{\geqslant 0} : \sum_{e \in S} x_e \leqslant r_1(S) \quad \forall S \subseteq E, \sum_{e \in S} x_e \leqslant r_2(S) \quad \forall S \subseteq E \right\}$$

   by showing that the above linear system is totally dual integral.

2. Let $Ax \leqslant b$ be a system of linear inequalities with A and b rational. Show that there exists a positive integer t such that $(1/t)Ax \leqslant (1/t)b$ is totally dual integral.

3. Show that the edge incidence matrix of an undirected graph $\mathcal{G}$ is totally unimodular if and only if $\mathcal{G}$ is a bipartite graph.

4. Show that the edge incidence matrix of any directed graph is a network matrix.

5. Show that the consecutive ones matrix is a network matrix.

6. Let $\mathcal{A}$ be a TU matrix. Show that duplicating a row or column preserves TU property.

7. Let $\mathcal{A}$ be a TU matrix. Show that negating a row or column preserves TU property.

8. Let $\mathcal{A}$ be a TU matrix. Show that adding a row or column with single 1 entry and all other entries being zero, preserves TU property.

9. Show that the coefficient matrix of the following linear programming formulation of the minimum-cost flow problem is totally unimodular.

   $$\begin{aligned} \text{minimize} \quad & p \cdot y \\ \text{s.t.} \quad & \forall v \in \mathcal{V}, x(\delta^+(v)) - x(\delta^-(v)) = b_v \\ & \forall e \in \mathcal{E}, x_e \leqslant c_e \\ & \quad x \geqslant 0 \end{aligned}$$

   Can we conclude from it that there is a integral optimal flow if all the demands (b vector) are integral?

10. Give an example of a polytope $\{x : Ax \leqslant b, x \geqslant 0\}$ which is integral but the matrix A is not TU.
11. Give an example of a polytope $\{x : Ax \leqslant b, x \geqslant 0\}$ which is integral and every entry of A is either 0 or ± 1 but the matrix A is not TU.