Practice Problems: Polytope, LP Duality, Complementary Slackness

Palash Dey
Indian Institute of Technology, Kharagpur

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Submit the solutions of the questions marked ($\star$) in PDF format generated using Latex by **February 20, 2025**.

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1. **($\star$)** In the Odd Cut problem, the input is an undirected edge-weighted graph $\mathcal{G}$ $(\mathcal{V},\mathcal{E})$ and we need to output a cut $(\mathcal{A}, \mathcal{V} \setminus \mathcal{A})$, if one exist, of minimum weight over all cuts whose both sides have an odd number of vertices. Design a polynomial-time algorithm for the Odd Cut problem. Observe that this algorithm gives a polynomial-time separation oracle for the perfect matching polytope for general graphs.

2. In the Even Cut problem, the input is an undirected edge-weighted graph $\mathcal{G}$ $(\mathcal{V},\mathcal{E})$ and we need to output a cut $(\mathcal{A}, \mathcal{V} \setminus \mathcal{A})$, if one exist, of minimum weight over all cuts whose both side have an even number of vertices. Design a polynomial-time algorithm for the Even Cut problem.

3. For a polyhedron $\mathcal{P}$ in $\mathbb{R}^n$, prove the following.

   (a) A point $v \in \mathbb{R}^n$ is a vertex of $\mathcal{P}$ if and only if v cannot be written as a convex combination of points in $\mathcal{P} \setminus \{v\}$.

   (b) A point $v \in \{x \in \mathbb{R}^n : Ax \leqslant b\}$ is a vertex if and only if there exists a subsystem $A'x \leqslant b'$ of $Ax \leqslant b$ such that $\{v\} = \{x \in \mathbb{R}^n : A'x = b'\}$ and the rank of A' is n.

   (c) A polytope is the convex hull of its vertices.

   (d) If $\mathcal{P}$ is a polytope, then for every vector $w \in \mathbb{R}^n$, there exists a vertex $\overline{x}$ of the polytope $\mathcal{P}$ which maximizes $w^T x$ among points $x \in \mathcal{P}$.

4. A *matchable set* of a graph $\mathcal{G}$ is the set of all vertices matched in some matching of $\mathcal{G}$. Let $\mathcal{P}$ be the convex hull of characteristic vectors of stable sets of a bipartite graph $\mathcal{G}$ $(\mathcal{V},\mathcal{E})$ with bipartition $\{A, B\}$. Prove that $\mathcal{P}$ is the same as the polytope defined by the following constraints. We denote the set of neighboring vertices of any $C \subseteq A$ by $N(C)$.

$$\left(\sum_{v\in C} x_v\right) - \left(\sum_{v\in N(C)} x_v\right) \leqslant 0, \forall C \subseteq A$$
$$x(A) - x(B) = 0$$
$$0 \leqslant x_v \leqslant 1, \forall v \in \mathcal{V}$$

5. A set of vertices of a graph is called a vertex cover if the set contains at least one end point of every edge. Show that the following polytope is the vertex cover polytope of a graph $\mathcal{G}$ $(\mathcal{V},\mathcal{E})$ if and only if $\mathcal{G}$ is a bipartite graph. Also show that the following polytope is half-integral. Write its dual linear program and complementary slackness conditions. Show integrality of this polytope for bipartite graphs using complementary slackness conditions.

$$\begin{aligned} \min \quad & \sum_{i\mathcal{V}} w_i x_i \\ \text{s.t.} \quad & x_u + x_v \geqslant 1 \qquad \forall\{u, v\} \in \mathcal{E}, \\ & 0 \leqslant x_i \leqslant 1 \qquad \forall i \in \mathcal{V}. \end{aligned}$$

6. Write the dual of the following linear programs and write the complementary slackness conditions.

   (a)
   $$\begin{aligned} \min \quad & 4x_1 + 3x_2 - x_3 \\ \text{s.t.} \quad & 2x_1 + x_2 \geqslant 5, \\ & x_1 + 3x_3 \geqslant 6, \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

   (b)
   $$\begin{aligned} \min \quad & 2x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 4, \\ & 2x_1 - x_2 \geqslant 1, \\ & x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

   (c)
   $$\begin{aligned} \max \quad & 3x_1 - x_2 \\ \text{s.t.} \quad & x_1 - x_2 \leqslant 5, \\ & 2x_1 + x_2 \geqslant 3, \\ & x_1 \geqslant 0 \end{aligned}$$

7. Write a linear programming formulation of maxumum $s - t$ flow in a directed edge-weighted graph. Write its dual and complementary slackness conditions.

8. Write a linear programming formulation for matching polytope of general graphs. Write its dual and complementary slackness conditions.