Problems: Stable Matching

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Note that the algorithm to compute a stable matching that was discussed during the class on September 3, 2026, is known by a few names. They are (i) Gale–Shapley Algorithm (ii) Gale–Shapley Deferred Acceptance Algorithm (iii) Men-Proposing Deferred Acceptance (DA) (iv) Proposer-Proposing Deferred Acceptance (v) Men-Proposing Gale–Shapley Algorithm (vi) Deferred Acceptance Procedure (DAP) (vii) Proposing Side DA.

1. [CLRS] Design/use a suitable data structure so that the men-proposing Gale-Shapley deferred acceptance algorithm runs in $\mathcal{O}(n^2)$ time.

2. [CLRS] The National Resident Matching Program differs from the scenario for the stable- marriage problem set out in this section in two ways. First, a hospital may be matched with more than one student, so that hospital h takes $r_h \geqslant 1$ students. Second, the number of students might not equal the number of hospitals. Describe how to modify the Gale-Shapley algorithm to ût the requirements of the National Resident Matching Program.

3. [CLRS] Prove the following property, which is known as weak Pareto optimality: Let M be the stable matching produced by the Gale-Shapley procedure, with women proposing to men. Then, for a given instance of the stable- marriage problem, there is no matching — stable or unstable — such that every woman has a partner whom she prefers to her partner in the stable matching M.

4. [CLRS] The stable-roommates problem is similar to the stable-marriage problem, except that the graph is a complete graph, not bipartite, with an even number of vertices. Each vertex represents a person, and each person ranks all the other people. The definitions of blocking pairs and stable matching extend in the natural way: a blocking pair comprises two people who both prefer each other to their current partner, and a matching is stable if there are no blocking pairs.

   Unlike the stable-marriage problem, the stable-roommates problem can have inputs for which no stable matching exists. Find such an input and explain why no stable matching exists.

5. Consider the men-proposing Deferred Acceptance algorithm for a stable marriage instance with n men and n women.

   Suppose that during the execution of the algorithm, a man m is rejected by a woman w.

   (a) Prove that m can never be matched with w in any stable matching.
   (b) Use the above result to prove that the matching produced by the men-proposing Deferred Acceptance algorithm is *men-optimal*; that is, every man weakly prefers his partner in the output matching to his partner in any other stable matching.

6. Consider a stable marriage instance with n men and n women.

   (a) Give an example of a preference profile having exactly one stable matching.
   (b) Give an example of a preference profile having more than one stable matching.
   (c) Investigate whether a stable marriage instance can have exponentially many stable matchings. State an appropriate bound and justify your answer.
   (d) Explain why enumerating all $n!$ perfect matchings is not an efficient algorithm for finding a stable matching.

7. Let M be a perfect matching between n men and n women.

(a) Design an $O(n^2)$-time algorithm to determine whether M is stable.

(b) Write pseudocode for your algorithm.

(c) Prove the correctness of your algorithm.

(d) Analyze its time and space complexity.