Practice Problems: Dynamic Programming for Designing Exponential Time Exact Algorithms

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Submit the solutions of the questions marked ($\star$) in PDF format generated using Latex by **March 28, 2025**.

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1. A set of arcs $\mathcal{F}$ in a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ is called a directed feedback arc set of $\mathcal{G}$ if $\mathcal{G}[\mathcal{V}, \mathcal{A} \setminus \mathcal{F}]$ is acyclic. Show that we can compute a minimum cardinality directed arc set of any directed graph in $\mathcal{O}^\star(2^n)$ time, where n is the number of vertices of $\mathcal{G}$.

2. **($\star$)** The cut-width of a vertex ordering π of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is

$$\max_{v \in \mathcal{V}} |\{\{w, x\} \in \mathcal{E} : \pi(w) \leqslant \pi(v) < \pi(x)\}|.$$

   The cut-width of a graph is the minimum cut-width taken over all orderings of its vertices. Show that we can compute the cut-width of a graph in time $\mathcal{O}^\star(2^n)$ time, where n is the number of vertices of $\mathcal{G}$.

3. The domatic number of a graph $\mathcal{G}$ is the minimum integer k such that $\mathcal{V}[\mathcal{G}]$ can be partitioned into k sets $\mathcal{V}_1, \ldots, \mathcal{V}_k$ such that each $\mathcal{V}_i$ is a dominating set of $\mathcal{G}$. Show how we can compute the domatic number of any graph in $\mathcal{O}^\star(3^n)$ time, where n is the number of vertices of $\mathcal{G}$.

4. In the EXACT SAT problem, we are given a CNF formula. The task is to check if there exists an assignment of its variables so that every clause has exactly one literal set to TRUE. Show that there is a $\mathcal{O}^\star(2^m)$ time algorithm for EXACT SAT, where m is the number of clauses in the formula.