Practice Problems: Edmond's Blossom Algorithm, Matroid

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Submit the solutions of the questions marked ($\star$) in PDF format generated using Latex by **January 31, 2025**.

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1 Edmond's Blossom Algorithm

1. **($\star$) Gallai-Edmond Decomposition:** The vertex set $\mathcal{V}$ of every graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ can be partitioned into three sets, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$. The set $\mathcal{D}$ consists of all vertices which are left unmatched in at least one maximum cardinality matching. They are called inessential vertices. The other vertices, i.e. the vertices in $\mathcal{V} \setminus \mathcal{D}$ are called essential vertices since they are matched in every maximum matching of $\mathcal{G}$. The set $\mathcal{A}$ consists of all essential vertices which are neighbor of at least one vertex in $\mathcal{D}$. The remaining set of vertices is called $\mathcal{B}$. Design an algorithm to construct the Gallai-Edmond decomposition of any given graph. (Hint: modify Edmond's blossom algorithm)

2 Matroids

Let $\mathcal{M} = (S, \mathcal{I})$ be any matroid.

2. Let $B \subset S$ be any subset of S and J a basis of B. We define another matroid $\mathcal{M}' = (S', \mathcal{I}')$ where $S' = S \setminus B$ and $\mathcal{I}' = \{J' \subseteq S' : J' \cup J \in \mathcal{I}\}$. Prove that $\mathcal{M}'$ is a matroid that does not depend of J and the rank of any $A \subseteq S'$ in $\mathcal{M}'$ is the rank of $A \cup B$ in $\mathcal{M}$ minus the rank of B in $\mathcal{M}$.

3. Let $J \in \mathcal{I}$ and $e \in S$ be any. Then $J \cup \{e\}$ contains at most one circuit.

4. A *branching* of a directed graph is a forest (of the underlying undirected graph) in which each edge has in-degree at most one. Suppose the edges of $\mathcal{G}$ are weighted. We want to compute a maximum weight branching of $\mathcal{G}$. Model this problem as a weighted matroid intersection problem.

5. Let $\mathcal{G}$ be a bipartite graph with $(\mathcal{L}, \mathcal{R})$ be a bipartition of the vertices. A transversal matroid $\mathcal{M}$ is defined on the set $\mathcal{L}$ where a subset $X \subseteq \mathcal{L}$ is an independent set if and only if X can be perfectly matched to a subset of $\mathcal{R}$. Show that the independence system defined above is a matroid.

6. Show that every uniform matroid is a transversal matroid.

7. **($\star$)** Show that every matching matroid is a transversal matroid.

8. Let $M = (S, \mathcal{I})$ be any matroid. Define the dual $M^\star = (S, \mathcal{I}^\star)$ of M as follows. The ground set of $M^\star$ is the ground set of M. A subset A of S is an independent set of $M^\star$ if and only if $S \setminus A$ contains a basis of M. Show that $M^\star$ is a matroid.

9. Let $(X_1, \ldots, X_k)$ be a partition of a set S and $d_i \geqslant 0$ for every $i \in [k]$. We say a set $A \subseteq S$ is independent if $|A \cap X_i| \leqslant d_i$ for every $i \in [k]$. Prove that this independence system is a matroid.

10. **(⋆)** Consider the following orientation problem. Given an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and integers $d_v \geqslant 0$ for every $v \in \mathcal{V}$, compute if it is possible to orient the edges of $\mathcal{G}$ such that the out-degree of every vertex $v \in \mathcal{V}$ is at most d_v in the resulting directed graph. Design a polynomial-time algorithm for this problem.

11. Suppose we are given an undirected graph $\mathcal{G}$ whose every edge has a (not necessarily distinct) color. Consider the problem of computing if the graph has a spanning tree whose all edges have different colors. Design a polynomial-time algorithm for this problem by reducing it to the matroid intersection problem.