Practice Problems on Markov Chains

1. Suppose we have a Monte Carlo algorithm $\mathcal{A}$ for some decision problem Π which, on input x, outputs the correct answer with probability $\geq 3/4$. Suppose $\mathcal{A}$ uses $O(\log n)$ random bits on inputs of size n. Prove that there exists a deterministic polynomial time algorithm for Π.

2. Consider drunken walk on integers $0, 1, 2, \ldots, n$ where the transition probabilities are
   - $\frac{1}{3}$ from k to $k+1$ for every $k \in [n-1]$
   - $\frac{2}{3}$ from k to $k-1$ for every $k \in [n-1]$
   - 1 from 0 to 1
   - n is an *absorbing state* i.e., once entered, the process never leaves state n.

   Prove that the expected number of steps to reach n from 0 is $\Omega(2^n)$. Can you give an example of a 3SAT formula where the 2SAT style randomised algorithm indeed takes $\Omega(2^n)$ steps?
   Hint: Think of a 3SAT formula with exactly one satisfying assignment.

3. Recall that the *lollipop graph* on n vertices consists of a clique on $n/2$ vertices connected to a path on $n/2$ vertices. Let u denote the vertex on the clique connected to the path and let v be the vertex at the other end of the path. Show that
   (a) the expected cover time of a random walk starting at v is $\Theta(n^2)$.
   (b) the expected cover time of a random walk starting at u is $\Theta(n^3)$.

4. Consider a finite Markov chain on n states with stationary distribution π and transition probabilities $(P_{ij})_{i,j \in [n]}$. Imagine starting the chain at time 0 and running it for m steps, obtaining the sequence of states $X_0, X_1, \ldots, X_m$. Consider the states in reverse order $X_m, X_{m-1}, \ldots, X_0$.
   (a) Argue that given X_{k+1}, the state X_k is independent of $X_{k+2}, X_{k+3}, \ldots, X_m$. Thus the reverse sequence is Markovian.
   (b) Argue that for the reverse sequence, the transition probabilities Q_{ij} are given by
   $$Q_{ij} = \frac{\pi_j P_{ji}}{\pi_i}.$$
   (c) Prove that if the original Markov chain is time reversible, so that $\pi_i P_{ij} = \pi_j P_{ji}$, then $Q_{ij} = P_{ij}$. That is, the states follow the same transition probabilities whether viewed in forward order or reverse order.

5. Show that the mixing time of a random walk on an n-dimensional hypercube is at most $n \ln n + n \ln \left(\frac{1}{\epsilon}\right)$.

6. Consider the following card shuffling process: insert the top card at a position chosen uniformly at random from 1 to n. Model the shuffling process as a Markov chain. Show that the unique stationary distribution is the uniform distribution and that $t_{\text{mix}}(\epsilon) \leq n \ln n + n \ln \left(\frac{1}{\epsilon}\right)$