Indian Institute of Technology Kharagpur
Department of Computer Science and Engineering

CS60029 Randomised Algorithm Design **Autumn 2025**

End-Semester Examination **Marks = 100** **22-Nov-2025, 2PM – 5PM**

*Answer **any five** questions. State all assumptions you make. Keep your answers concise.*

If not mentioned otherwise, assume that n and m are any positive integers at least 2, and p is any prime number at least 2.

1. (a) Prove that there is no algorithm with polylogarithmic (in n and m) space complexity that outputs the frequency of any element queried at the end of an m length stream over a universe of size n.

   (b) Prove or disprove: The hash family $\{h_{a,b} : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_2$, defined as $h_{a,b}(x) = ax + b \pmod 2 : a, b \in \mathbb{Z}_{10}, a \neq 0\}$ is a 2-universal hash family.

   8+12 = 20

2. Let $\{X_i\}_{i=1}^n$ be a sequence of zero-mean random variables (not necessarily independent), each sub-Gaussian with parameter σ. Prove that

   (a) $\mathbf{E}[\max_{i \in [n]} X_i] \leq \sqrt{2\sigma^2 \log n}$

   Hint: The logarithmic function is concave and exponential function is convex.

   (b) $\Pr[\max_{i \in [n]} X_i \geq t] \leq ne^{-t^2/2\sigma^2}$

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3. The Monte Carlo method for estimating the value of π discussed in the class assumes that we can draw uniform samples from a 2×2 square in $\mathbb{R}^2$, which is an uncountably infinite set. Design an (ϵ, δ)-approximator of π, which needs to draw uniform samples from finite sets only. In particular, it does not need to draw sample from any infinite set. 20

4. Use the probabilistic method for the following.

   (a) Let F be a finite collection of binary strings of finite lengths and assume no member of F is a prefix of another one. Let n_i denote the number of strings of length i in F. Prove that

   $$\sum_i \frac{n_i}{2^i} \leq 1.$$

   (b) Let $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ with $\|\vec{v}_i\| = 1$ for all $i \in [n]$ (here, $\|\vec{v}\|$ denotes the L_2-norm of $\vec{v}$). Show that there exist $e_1, \ldots, e_n \in \{-1, 1\}$ such that

   $$\|e_1\vec{v}_1 + \cdots + e_n\vec{v}_n\| \geq \sqrt{n}.$$

   10+10 = 20

5. Show that it is possible to colour the edges of K_n (complete graph on n vertices) so that it has no monochromatic K_k-subgraph if

   $$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1.$$

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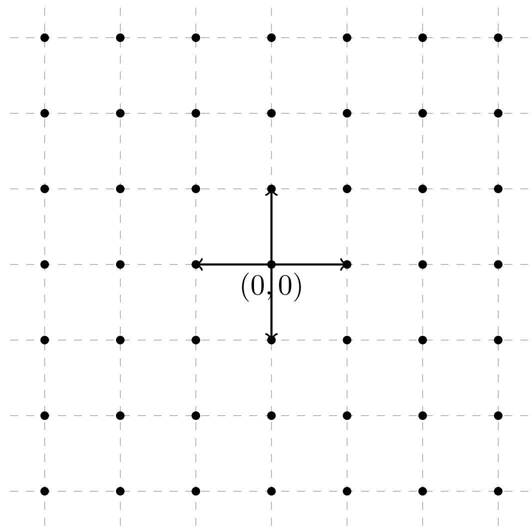
6. Let S_n be a simple symmetric random walk on the square lattice $\mathbb{Z}^2$ with $S_0 = (0,0)$. The walker starts from the origin and at each step, independently moves one unit to East, North, West, South directions with equal probability $(1/4)$. Let D_n be the walker's Euclidean distance from the origin at time n (i.e., after n moves).



Which one of the following sequences is a martingale? Justify your answer.

(a) $\{D_n\}_{n\geq 0}$

(b) $\{D_n^2 - n\}_{n\geq 0}$

(c) $\{D_n - n\}_{n\geq 0}$

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