Solutions to CS60029 Randomized Algorithm Design
Class Test 1

Q1. Triangle Packing via Color Coding (15 points)

We are given a graph $G = (V, E)$ and an integer k. We want to test whether there exist k vertex-disjoint triangles.

Step 1: Random Coloring

Assign each vertex a color from $\{1, 2, \ldots, 3k\}$ uniformly at random. A solution uses exactly $3k$ vertices. If these vertices all get distinct colors, we call it a *colorful solution.*

Step 2: Probability of Success

For any fixed set of $3k$ vertices, the probability that they all receive distinct colors is

$$\frac{(3k)!}{(3k)^{3k}} \geq e^{-3k}.$$

Repeating the experiment $2^{O(k)}$ times ensures success with high probability.

Step 3: Detecting Colorful Triangles

For each triangle, check if its vertices have distinct colors. Then, search for k disjoint colorful triangles. This reduces to a dynamic programming problem:

- State: subset of colors used, number of triangles formed.
- Transition: add a new colorful triangle disjoint in colors.

This DP runs in $O(2^{3k} \cdot n^{O(1)})$.

Q2(a). Necessity of Non-Negativity in Markov's Inequality (5 points)

Markov's inequality: For a non-negative random variable X and $a > 0$,

$$\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$

Counterexample if X can be negative:

$$X = \begin{cases} -1 & \text{with prob. } 0.9, \\ 9 & \text{with prob. } 0.1. \end{cases}$$

Then $\mathbb{E}[X] = 0$. For $a = 5$,

$$\Pr[X \geq 5] = 0.1, \quad \frac{\mathbb{E}[X]}{a} = 0.$$

Thus Markov's inequality fails. Hence, the non-negativity assumption is necessary.

Q2(b). Weak Law of Large Numbers using Chebyshev (10 points)

Let $X_1, X_2, \dots$ be i.i.d. random variables with mean μ and variance σ^2. Define

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Step 1: Expectation and Variance

$$\mathbb{E}[\bar{X}_n] = \mu, \quad \mathrm{Var}(\bar{X}_n) = \frac{\sigma^2}{n}.$$

Step 2: Apply Chebyshev

By Chebyshev's inequality,

$$\Pr\left(|\bar{X}_n - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2}.$$

Step 3: Limit

As $n \to \infty$,

$$\frac{\sigma^2}{n\epsilon^2} \to 0.$$

Hence,

$$\lim_{n\to\infty} \Pr\left(|\bar{X}_n - \mu| > \epsilon\right) = 0.$$

This proves the Weak Law of Large Numbers.