# Practice Problems: Matroid

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Let $\mathcal{M} = (S, \mathcal{I})$ be any matroid.

1. Let $B \subset S$ be any subset of $S$ and $A \subset S \setminus A$ any subset of $S \setminus A$. Prove that the rank of $A$ in $\mathcal{M}'$ is the rank of $A \cup B$ in $\mathcal{M}$ minus the rank of $B$ in $\mathcal{M}$.
2. Let $J \in \mathcal{I}$ and $e \in S$ be any. Then $J \cup \{e\}$ contains at most one circuit.
3. A *branching* of a directed graph is a forest (of the underlying undirected graph) in which each edge has in-degree at most one. Suppose the edges of $\mathcal{G}$ are weighted. We want to compute a maximum weight branching of $\mathcal{G}$. Model this problem as a weighted matroid intersection problem.
4. Let $\mathcal{G}$ be a bipartite graph with $(\mathcal{L}, \mathcal{R})$ be a bipartition of the vertices. A transversal matroid $\mathcal{M}$ is defined on the set $\mathcal{L}$ where a subset $X \subseteq \mathcal{L}$ is an independent set if and only if $X$ can be perfectly matched to a subset of $\mathcal{R}$. Show that the independence system defined above is a matroid.
5. Show that every uniform matroid is a transversal matroid.
6. Let $(X_1, \ldots, X_k)$ be a partition of a set $S$ and $d_i \geqslant 0$ for every $i \in [k]$. We say a set $A \subseteq S$ is independent if $|A \cap X_i| \leqslant d_i$ for every $i \in [k]$. Prove that this independence system is a matroid.
7. Consider the following orientation problem. Given an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and integers $d_v \geqslant 0$ for every $v \in \mathcal{V}$, compute if it is possible to orient the edges of $\mathcal{G}$ such that the out-degree of every vertex $v \in \mathcal{V}$ is at most $d_v$ in the resulting directed graph. Design a polynomial-time algorithm for this problem.