Probabilistic Method

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1. Prove or disprove: every tournament graph has a Hamiltonian path.

2. A k-uniform hypergraph is a graph $G = (V, E)$ such that each edge $e \in E$ is a k-tuple $(v_1, \ldots, v_k)$. Suppose we color each vertex of a k-uniform hypergraph G one of two color. Let c_k be the smallest integer such that if G has c_k edges then it has at least one monochromatic edge. Show that $c_k \geqslant 2^{k-1}$.

3. A family $\mathcal{F}$ of sets is *intersecting* if for all $A, B \in \mathcal{F}, A \cap B \neq \emptyset$. Prove that if $|X| = n, n \geqslant 2k$, and $\mathcal{F}$ is an intersecting family of k-element subsets of X, then

   $$|\mathcal{F}| \leqslant \binom{n-1}{k-1}.$$

   This is known as the Erdos-Ko-Rado Theorem.

4. Show using method of expectation that every graph with m edges has a bipartite subgraph with at least $\frac{m}{2}$ edges.

5. For any $k, \ell > 0$, there exists a graph with chromatic number more than k and the length of the shortest cycle being more than ℓ.

6. Let G be a k-uniform hypergraph in which every hyperedge intersect at most $\frac{2^{k-1}}{e} - 1$ other hyperedges. Then show that G is 2 colorable.