Indian Institute of Technology Kharagpur

CS31005 Algorithms II – Class Test 1 Total marks: 30 Duration: 1 hr

Name: ____________________

Roll Number: ____________________

Answer all questions.

1. (10 points) We are given a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with weights $w : \mathcal{E} \longrightarrow \mathbb{R}_{\geqslant 0}$ on edges. Assume $\mathcal{G}$ does not contain any anti-parallel edges. Suppose that you wish to find, among all minimum $s - t$ cuts in a flow network $\mathcal{G}$ with integral capacities, one that contains the smallest number of edges. Show how to modify the capacities of $\mathcal{G}$ to create a new flow network $\mathcal{G}'$ in which any minimum $s - t$ cut in $\mathcal{G}'$ is a minimum $s - t$ cut with the smallest number of edges in $\mathcal{G}$.

2. (a) (5 points) Consider an ordinary balanced binary search tree data structure with n elements supporting the instructions INSERT, DELETE-MIN, and DELETE-MAX in $\mathcal{O}(\log n)$ worst-case time. Show that the amortized cost of INSERT is $\mathcal{O}(\log n)$ and both DELETE-MIN and DELETE-MAX are $\mathcal{O}(1)$.

   (b) (5 points) Draw the Fibonacci heaps after performing each of the following operations in an empty Fibonacci heap: insert(10), insert(20), insert(40), extract-min, insert(15), insert (50), extract-min, extract-min, insert(10), extract-min.

3. (a) (10 points) Suppose for a flow graph $G = (V, E)$ when we run the Edmond Karp algorithm to find a max flow then upon termination of the algorithm the set of edges with positive flow is $E_{ek} \subseteq E$, and when we run the Push Relabel algorithm to find max flow then upon termination of the algorithm the set of edges with positive flow is $E_{pr} \subseteq E$. For each $n \geqslant 5$, design a flow graph $G = (V, E)$ such that $|V| = n$, for every vertex v, there is an s-t path passing through v in G, every edge has positive capacity, and $E_{ek} \neq E_{pr}$. For $n = 6$, show all steps of execution of the Edmond Karp algorithm and the Push Relabel algorithm on such a G that results in different E_{ek} and E_{pr}.

4. Consider the problem of insertions and deletions of positive integers into a table that is solved by the following algorithm $\mathcal{A}$: Let the current table size be i.

   ▷ Elements are inserted into the current table till the table is full. When we try to insert an element but the table already has i elements, a new table of size $2i$ is created and all elements including the new element are inserted into the new table while the old table is forgotten.

   ▷ Elements may be deleted from the current table upto the point where the table contains $i/4$ elements. When an element has to be deleted but the table has $i/4$ elements, then a new table of size $i/4$ is created and all elements minus the element to the currently deleted are copied onto the new table while the old table is forgotten.

   (a) (5 points) Analyse the amortized running time of an insertion operation in the algorithm $\mathcal{A}$.

   (b) (5 points) Analyse the amortized running time of a deletion operation in the algorithm $\mathcal{A}$.

5. Consider the problem Area-Union, that takes as input a set of axis-parallel rectangles and outputs the area of the union of these rectangles. A rectangle is axis-parallel if each side is parallel to the x-axis or y-axis. **Also assume that your input instance is such that any rectangle can intersect with at most one other rectangle.**
   (a) (5 points) Design a sweep line algorithm for Area-Union. [Hint: What should be the events of the sweep line? What area is covered between events?]
   (b) (5 points) What is the running time of the algorithm?