INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR
Algorithmic Game Theory 2021-22: Sample Solution Sketch of First Class Test

The last two digits of your roll number from right be d_2 and d_1 respectively. Let us define $d = 10 * d_2 + d_1$.
There are two farms, namely A and B, who produce the same goods. If farm A (respectively B) produces q_A (respectively q_B) quantity of goods, the total goods q available in the market is $q_A + q_B$. The price p per unit is given by the following formula.

$$p(q) = \begin{cases} 140 - q & \text{if } q < 140 \\ 0 & \text{otherwise} \end{cases}$$

The cost of production is d per unit of goods for both the farms. Hence, the profit for farm A and B are $(q_A * p(q) - d * q_A)$ and $(q_B * p(q) - d * q_B)$ respectively. Each farm needs to independently decide how much quantity (which are non-negative real numbers) of goods to produce.

1. Model the above situation as a two player normal form game.

[2 Marks]

For concreteness, let us assume that $d = 20$. The situation can be modeled by the strategic form game $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$ where

- ▷ $N = \{A, B\}$
- ▷ $S_A = \mathbb{R}_{\geqslant 0}, S_B = \mathbb{R}_{\geqslant 0}$
- ▷ $u_A(q_A, q_B) = p(q_A + q_B) * q_A - 20 * q_A, u_B(q_A, q_B) = p(q_A + q_B) * q_B - 20 * q_B$

2. Compute a PSNE for this game.

[6 Marks]

Let $(q_A^*, q_B^*) \in S_A \times S_B$ be a PSNE. From the definition of PSNE, the function $u_A(q_A, q_B^*) = p(q_A + q_B^*) * q_A - 20 * q_A$ where only q_A varies, is maximized at $q_A = q_A^*$ in the domain $\mathbb{R}_{\geqslant 0}$. Hence, we have

$$\left(\frac{\partial u_A(q_A, q_B^*)}{\partial q_A}\right)_{q_A = q_A^*} = 120 - 2q_A^* - q_B^*$$

Similarly, we have

$$\left(\frac{\partial u_A(q_A^*, q_B)}{\partial q_B}\right)_{q_B = q_B^*} = 120 - 2q_B^* - q_A^*$$

From equations $\left(\frac{\partial u_A(q_A, q_B^*)}{\partial q_A}\right)_{q_A = q_A^*} = 0$ and $\left(\frac{\partial u_A(q_A^*, q_B)}{\partial q_B}\right)_{q_B = q_B^*} = 0$, we obtain $q_A^* = q_B^* = 40$. Hence, $(40, 40)$ is a PSNE of Γ. The utility of both the players in the strategy profile $(40, 40)$ is 1600.

3. Compute a strategy profile where both farms receive higher utility than your computed PSNE.

[2 Marks]

In the strategy profile $(30, 30)$, both the farms receive a utility of 1800 each.

1. Compute all MSNEs of the following game.

   ▷ The set of players $(N) : \{1, 2\}$

   ▷ The set of strategies: $S_i = \{A, B\}$ for every $i \in [2]$

   ▷ Payoff matrix:

		Player 2	
		A	B
Player 1	A	(d^2, d^2)	$(0, 0)$
	B	$(0, 0)$	(d, d)

[6 Marks]

Let us assume that $d = 5$. Clearly, (A, A) and (B, B) are PSNEs of this game. Let $(x = (x_A, x_B), y = (y_A, y_B))$ be an MSNE of this game. If $x_A = 0$, then we have $u_2(x, B) > u_2(x, A)$ and thus we have $y_A = 0$. Similarly, if $x_B = 0$, then $y_B = 0$; if $y_A = 0$, then $x_A = 0$; if $y_B = 0$, then $x_B = 0$. Hence, let us assume without loss of generality that we have $x_A, x_B, y_A, y_B > 0$. From indifference principle, we have

$$\begin{aligned} & u_1(A, y) = u_1(B, y) \\ \Rightarrow & 25y_A = 5y_B \\ \Rightarrow & 5y_A = y_B \end{aligned}$$

and

$$\begin{aligned} & u_2(x, A) = u_2(x, B) \\ \Rightarrow & 25x_A = 5x_B \\ \Rightarrow & 5x_A = x_B \end{aligned}$$

Solving the above equations along with $x_A + x_B = 1$ and $y_A + y_B = 1$, we obtain $x_A = \frac{1}{6}, x_B = \frac{5}{6}, y_A = \frac{1}{6}, y_B = \frac{5}{6}$.

2. Prove or disprove: there exists a game with d weakly dominant strategy equilibriums.

[4 Marks]

If $d = 0$, then the coordination game above does not have any weakly dominant strategy equilibrium.
If $d = 1$, then the second price selling auction has unique weakly dominant strategy equilibrium.
If $d > 2$, then there does not exist any game with d WDSEs. Indeed, a WDSE, if exists, is unique. Proof is left as homework.