Assignment 4: Randomized Algorithm Design

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1. Let C_n be a cycle on a set $\mathcal{V}$ of n vertices and $\mathcal{T}$ be a tree which is an embedding of C_n. Then prove that there exists an edge $\{u, v\} \in \mathcal{E}[C_n]$ such that the distance between u and v in $\mathcal{T}$ is $n - 1$.

2. If $Z_i, i \in \mathbb{N}$ is a martingale with respect to $X_i, i \in \mathbb{N}$, then prove that $Z_i, i \in \mathbb{N}$ is a martingale with respect to itself also.

3. Let $X_0 = 0$ and X_{j+1} is distributed uniformly over $[X_j, 1]$. Show that, for $k \geqslant 0$, the sequence

$$Y_k = 2^k(1 - X_k)$$

   is a martingale.

4. Alice and Bob play each other in a checkers tournament, where the first player to win four games wins the match. The players are evenly matched, so the probability that each player wins each game is $\frac{1}{2}$, independent of all other games. The number of minutes for each game is uniformly distributed over the integers in the range $[30, 60]$, again independent of other games. What is the expected time they spend playing the match?

5. Consider an urn that initially contains b black balls and w white balls. At every iteration, we draw a random ball is chosen and the chosen ball is replaced by $c > 1$ balls of the same color. Let X_i denote the fraction of black balls after i-th draw. Prove that $X_0, X_1, \ldots$ is a martingale.