Assignment 3: Randomized Algorithm Design

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April 2, 2020

1. A set family $\mathcal{F} \in 2^{[n]}$ is called *intersecting* if, for every $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, we have $\mathcal{A} \cap \mathcal{B} \neq \emptyset$. Let $\mathcal{F}$ be an intersecting family of subsets of cardinality k of $[n]$. Then prove that $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$. This is known as the Erdős-Ko-Rado Theorem or the sunflower theorem.

2. Prove that any graph has a bipartite sub-graph containing at least half the total number of edges.

3. The chromatic number $\chi(G)$ of a graph G is the minimum number of colors needed to color its vertices. The girth $g(G)$ is the lenght of the shortest cycle in the graph G (if there is no cycle, then the girth is -1). Prove that, for any $k, \ell > 0$ there exists a graph with $\chi(G) > k$ as well as $g(G) > \ell$.

4. A family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, n\}$ is called an anti-chain if, for no pair of sets $A, B \in \mathcal{F}$, we have $A \subseteq B$.

   (a) Give an example of $\mathcal{F}$ with $\mathcal{F} = \binom{n}{\lfloor n/2 \rfloor}$.

   (b) Let f_k denote the number of sets in $\mathcal{F}$ of cardinality k. Then prove

   $$\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leqslant 1$$

   (c) Prove that $|\mathcal{F}| \leqslant \binom{n}{\lfloor n/2 \rfloor}$ for any anti-chain $\mathcal{F}$.

5. An hypergraph is a tuple $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of hyperedges. An hyperedge $e \subseteq \mathcal{V}$ is some subset of vertices. A coloring of a hypergraph gives colors to its vertices such that no hyper edge sees vertices of exactly one color. Prove that, in a hypergraph $\mathcal{H}$, if every edge has at least k vertices, every edge intersects with at most d other edges, and $e(d+1) \leqslant 2^{k-1}$, then $\mathcal{H}$ is 2 colorable. (Hint: use Lovász local lemma)

6. Let $\mathcal{D} = (\mathcal{V}, \mathcal{A})$ is a directed graph with minimum outdegree δ and maximum indegree Δ. Prove (using Lovász local lemma) that, if $k \leqslant \frac{\delta}{1+\ln(1+\delta\Delta)}$, then $\mathcal{D}$ contains a directed cycle of length divisible by k.

7. The randomized algorithm of assigning every vertex to one of the two sets A and B with equal probability provides a $1/2$ factor approximation for the maximum cut problem for weighted undirected graphs. Use method of conditional expectation to de-randomize this algorithm.

8. Give an example of a 2 universal hash family which is not 3 universal.

9. In the standard balls and bins setting, we have seen that if n balls are thrown into n bins uniformly at random, then the maximum load is at most $2\ln n / \ln\ln n$ with probability at least $1 - 1/n$. Suppose, we use a hash function picked u.a.r from a k-universal hash family to throw balls into bins. Show that, for $k = 2$, the maximum load is at most $\sqrt{2n}$ with probability at least $1/2$. Find the smallest value of k such that the maximum load is at most $2\ln n / \ln\ln n$ with probability at least $1/2$.